5

Event-Based Control

L. Grüne, S. Hirche, O. Junge, P. Koltai, D. Lehmann, J. Lunze, A. Molin, R. Sailer, M. Sigurani, C. Stöcker, and F. Wirth

In event-based control, the feedback loop is closed only if an event indicates that the control error exceeds a tolerable bound and triggers a data transmission from the sensors to the controllers and the actuators. Hence, event-based control is an important method for reducing the communication load of a digital network. This chapter explains the main ideas of event-based control and proposes new loop structures and design methods.

Chapter Contents

5.1 Introduction to Event-Based Control

Event-based control is a control methodology that is currently being developed as a means to reduce the communication between the sensors, the controller and the actuators in a control loop. The sampling instants are not determined periodically by a clock, but by an event generator, which adapts the information flow in the feedback loop to the current behavior of the closed-loop system. A communication among the components is invoked only after an event has indicated that the control error exceeds a tolerable bound.

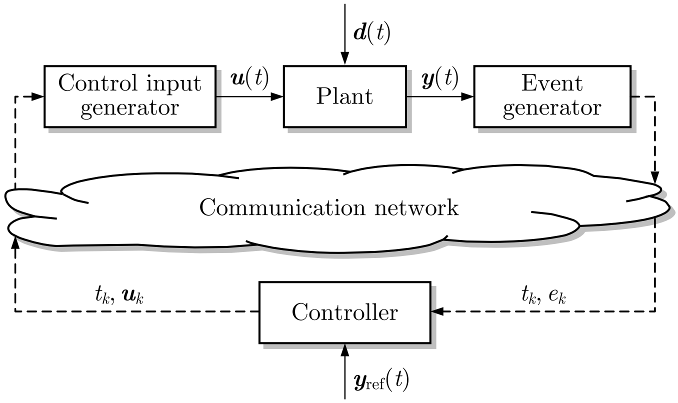

Fig. 5.1 Event-based control loop

This working principle differs fundamentally from that of sampled-data feedback loops, in which the sensor data are communicated to the controller at equidistant sampling times. In periodic sampling, a communication takes place independently of the size of the control error and, in particular, also in case of small control errors when an information feedback is not necessary to satisfy the performance requirements on the closed-loop system. In these situations, the communication and computing resources are used unnecessarily.

Figure 5.1 shows the main components of an event-based control loop. The plant has the continuous-time or discrete-time input $\boldsymbol{u}(t)$ and the state $\boldsymbol{x}(t)$ (or output $\boldsymbol{y}(t)$), which are continuously generated by the control input generator or continuously evaluated by the event generator, respectively. The communication links drawn by dashed arrows are only used after the event generator has indicated that the control error exceeds a tolerable bound at some time t_k, where an event name e_k, the current state $\boldsymbol{x}(t_k)$ or the current output $\boldsymbol{y}(t_k)$ is transmitted to the controller. The controller determines the new input $\boldsymbol{u}_k$, which is used by the control input generator to determine the continuous input $\boldsymbol{u}(t)$ in the time interval $[t_k, t_{k+1})$ until the next event occurs at time t_{k+1}.

Theoretical Challenges. In event-based control, the fundamental assumption of sampled-data control theory claiming a periodic triggering scheme invoked by a clock is violated. Hence, for event-based control the well-known discrete-time models of the plant, the controller and the closed-loop system cannot be applied but a new theory has to be developed, which takes into account the asynchronous component behavior. The main novel analysis and design aims of this new theory refer to the choice of the event generator and of the control input generator.

The **event generator** determines

- the time instants t_k, $(k = 0, 1, ...)$ at which the next communication between the event generator, the controller and the control input generator is invoked, and
- the information that is communicated from the sensor towards the controller.

The **control input generator** determines the signal $\boldsymbol{u}(t)$ continuously for the time interval $t \in [t_k, t_{k+1})$ in dependence upon the information obtained at time t_k.

The **main questions** to be answered ask

- at which time t_k a feedback loop has to be closed by using the communication links,
- which information e_k should be communicated and
- how the control input generator should determine the control input $\boldsymbol{u}(t)$ between succeeding event times.

Several different methods for event-based control have been proposed in the recent years, which distinguish with respect to the answers given to these questions. Some of them have been published under different names like *event-driven control*, *event-triggered control*, *Lebesgue sampling*, *deadband control* or *send-on-delta control* . Surveys and introductions to these techniques can be found in [241, 354].

A similar, but conceptually different methodology is *self-triggered control.* Here the plant state is not continuously supervised by the event generator, but the next event time t_{k+1} is determined by the event generator at the event time t_k [6, 379]. Then the sensors can "sleep" until the predicted next sampling instant.

Application Scenarios. There are several reasons for using event-based feedback. First, the information exchange in the feedback loop should be reduced to the minimum communication that is necessary to ensure a required system performance. If the information is transferred to and from the controller by a digital communication network, a reduced information flow decreases the risk of a network overload. For wireless nodes, reduced activity saves energy.

The second motivation occurs for systems, where the physical structure requires that measurements or control actions have to be taken at time instants prescribed by the dynamics of the plant. For example, position measuring devices of rotating components work with markings on the axis, which provide position information not at specific clock times but in certain positions.

Third, asynchronous communication protocols and real-time software do not allow to transfer and process information at specific clock times but lead to an inherently asynchronous behavior of all components of a feedback loop. Sensors and actuators likewise are triggered by events, because they work if some new information arrives. For such components the event-based working scheme is more "natural" than periodic sampling.

Fundamental Properties of Event-Based Control. The event-driven function principle implies that the plant input $\boldsymbol{u}(t)$ is determined by a combination of feedforward control and feedback control. At event times t_k, the input $\boldsymbol{u}(t_k)$ depends in a closed-loop fashion upon the current state $\boldsymbol{x}(t_k)$ (provided that the communication links do not introduce a substantial time delay), whereas between two consecutive event times t_k and t_{k+1} the input $\boldsymbol{u}(t)$ is generated as open-loop control in dependence upon "old data" $\boldsymbol{u}_k$.

Due to the aim of event-based control to sample only if a severe performance degradation has to be avoided, most event-based control schemes cannot ensure asymptotic stability of the closed-loop system. Instead, the plant state $\boldsymbol{x}(t)$ should be held in the surroundings Ω of the equilibrium state $\bar{\boldsymbol{x}}$. The property $\boldsymbol{x}(t) \in \Omega$ for all $t \geq \bar{t}$ is called *ultimate boundedness* or *practical stability* of the closed-loop system (cf. Def. 5.1). Usually the size of the set Ω depends upon the event threshold $\bar{e}$.

Comparison of Event-Based and Sampled-Data Control. Analytical results showing the difference of event-based control and sampled-data control can only be obtained for first-order systems [11, 13, 226, 303]. They show that event-based sampling can lead indeed to a considerable reduction of the communication within the control loop. Furthermore, if the system is heavily disturbed, the event-based control loop may have a better performance than the sampled-data loop, because in this situation it invokes the communication more often than the clock.

Chapter Overview. This chapter surveys methods that are applicable for high-order systems and different event-triggering and control input generation methods. Five different methods will be described in the sequel:

- **Event-based state feedback** (Section 5.2): The control input generator is constructed such that the closed-loop system mimics the behavior of a continuous state-feedback loop with adjustable precision. It is shown that the control input generator has to include a model of the continuous state-feedback loop.

- **Distributed event-based control** (Section 5.3): For interconnected systems, the idea of event-based state feedback can either be implemented as a distributed controller, which leads to the same overall system performance as the centralized feedback, or can be applied separately to the isolated subsystens resulting in a decentralized event-based control scheme.

- **Optimization-based control** (Section 5.4): The event-based controller can be obtained as the solution of an optimal control problem, if the state space is partitioned and the best possible constant input is applied as long as the state remains in the same state-space partition.

- **Event-based stabilization of large-scale systems** (Section 5.5): The stability of an interconnected system is tested by a small-gain theorem, which also evaluates the robustness of the event-based control loop against uncertainties in the communication channel.

- **Event-based control of interconnected nonlinear systems** (Section 5.6): This section extends the idea of event-based control to nonlinear interconnected systems and proves that robustly stable controllers of the subsystems lead to an input-to-state stable overall system.

- **Event-based control of stochastic systems** (Section 5.7): If formulated as a stochastic optimization problem, the event generator and the control input generator have to be designed simultaneously leading to a very complex optimization problem. If, however, both components lie in a nested information structure, where the event generator knows the plant state $\boldsymbol{x}(t)$ for all time t and the control input generator knows only the state $\boldsymbol{x}(t_k)$ at an event time t_k, the overall problem can be decomposed into two nested optimization problems that can be solved sequentially.

There is an interesting similarity concerning the structure of the event-based controllers that are developed in Sections 5.2 and 5.7 by starting from quite different viewpoints. Section 5.7 looks for optimal event-triggering and input-generating policies and ends up with an event generator and an control input generator that include a model of the closed-loop system. This structure is quite similar to the control loop elaborated in Section 5.2, which is obtained by investigating the control law of a state-feedback controller and implementing it in an event-based fashion. Likewise, the event generator and the control input generator have to include a model of the continuous control loop.

The methods developed in Sections 5.3 and 5.6 have been experimentally evaluated at a common thermofluid process, which is described in Section 5.8.

5.2 Disturbance Attenuation by Event-Based State Feedback

5.2.1 Control Aim

This section focusses on the influence of unknown disturbances on event-based control. It introduces a state-feedback approach to event-based control with the design aim to mimic the disturbance behavior of a given continuous closed-loop system, which is assumed to have the "best possible" disturbance attenuation properties.

In the sequel, the event-based control loop shown in Fig. 5.1 is simplified as depicted in Fig. 5.2. Here, the controller is incorporated in the control input generator.

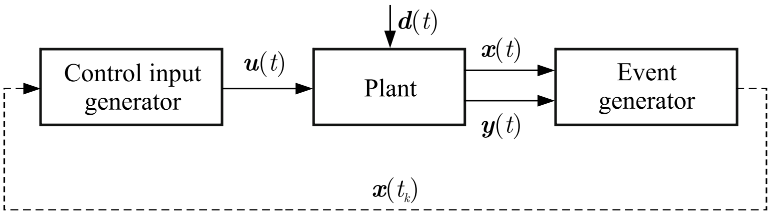

Fig. 5.2 Simplified event-based control loop

The plant is represented by the linear state-space model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{5.1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \tag{5.2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ denotes the state of the system with the initial value $\boldsymbol{x}_0$, $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^r$ are the inputs or measured outputs, respectively, and $\boldsymbol{d} \in \mathbb{R}^l$ represents exogenous disturbances.

The pair $(\boldsymbol{A}, \boldsymbol{B})$ is assumed to be controllable and the disturbance $\boldsymbol{d}(t)$ to be bounded:

$$\|\boldsymbol{d}(t)\| \leq d_{\max}. \tag{5.3}$$

The notations $\|\boldsymbol{x}\|$ and $\|\boldsymbol{A}\|$ denote an arbitrary vector norm or the induced matrix norm, and the absolute value is denoted by $|x|$. The expression $\|\boldsymbol{x}(t)\|$ denotes an arbitrary vector norm at time t. It is further assumed that

- the plant dynamics are accurately known,
- the state $\boldsymbol{x}(t)$ is measurable, and
- the information exchange between the event generator and the control input generator is instantaneous and imposes no restrictions on the information to be sent at event times.

jan.lunze@ruhr-uni-bochum.de

Hence, the reason to communicate information via the dashed arrows in Fig. 5.2 is primarily given by the situation that the disturbance $\boldsymbol{d}(t)$ has caused an intolerable behavior of the control output $\boldsymbol{y}(t)$ or the plant state $\boldsymbol{x}(t)$.

Main Idea. As a main characteristic of the scheme proposed, the event generator uses a model of the continuous control loop to compare the current plant state $\boldsymbol{x}(t)$ with the desired state that occurs in the continuous closed-loop system. If the difference between both states exceeds an upper bound $\bar{e}$, an event is triggered and the current state $\boldsymbol{x}(t_k)$ is transmitted to the control input generator. As a further important fact, the control input generator incorporates the model of the continuous control loop to determine the future control input $\boldsymbol{u}(t)$, $(t \geq t_k)$. It will be shown that the event-based control loop with these characteristics has the following properties:

- The state $\boldsymbol{x}(t)$ of the event-based state-feedback loop is ultimately bounded in the sense that it remains, for all times t, in a bounded neighborhood Ω_{e} of the desired state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous state-feedback loop.
- The communication over the feedback channel in the event-based control loop is bounded and depends explicitly on the disturbance $\boldsymbol{d}(t)$.
- Both the accuracy in terms of approximating the behavior of the continuous state-feedback loop and the minimum time interval between two consecutive events (*minimum inter-event time*) can be adjusted by changing the threshold $\bar{e}$ of the event generator in order to adapt the event-based state-feedback loop to the requested needs.

5.2.2 Continuous State Feedback

This section summarizes the main properties of the continuous state-feedback loop which is later used as the reference system to evaluate the behavior of the event-based state-feedback loop. Plant (5.1), (5.2) together with the state feedback

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) \tag{5.4}$$

yields the continuous closed-loop system

$$\dot{\boldsymbol{x}}_{\mathrm{CT}}(t) = \underbrace{(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})}_{\bar{\boldsymbol{A}}}\boldsymbol{x}_{\mathrm{CT}}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}_{\mathrm{CT}}(0) = \boldsymbol{x}_0 \tag{5.5}$$

$$\boldsymbol{y}_{\mathrm{CT}}(t) = \boldsymbol{C}\boldsymbol{x}_{\mathrm{CT}}(t). \tag{5.6}$$

The index "CT" is used to distinguish the signals of this model from the corresponding signals of the event-based control loop considered later.

jan.lunze@ruhr-uni-bochum.de

The state-feedback matrix $\boldsymbol{K}$ is assumed to be designed so that the matrix $\bar{\boldsymbol{A}}$ is Hurwitz and the closed-loop system has desired disturbance attenuation properties.

As $\bar{\boldsymbol{A}}$ is Hurwitz and the disturbance $\boldsymbol{d}(t)$ is assumed to be bounded according to Eq. (5.3), the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous state-feedback loop (5.5), (5.6) is *GUUB* according to the following definition:

Definition 5.1. [205] *The solution $\boldsymbol{x}(t)$ of the continuous control loop* (5.5), (5.6) *is said to be globally uniformly ultimately bounded (GUUB) if for every $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists a positive constant p and a time $\bar{t}$ such that holds:*

$$\boldsymbol{x}(t) \in \Omega_{\mathrm{t}} = \{\boldsymbol{x} : \|\boldsymbol{x}\| \leq p\}, \quad \forall t \geq \bar{t}.$$

Then one says that the continuous control loop (5.5), (5.6) is ultimately bounded.

For the linear continuous control loop (5.5), (5.6) the state $\boldsymbol{x}(t)$ is GUUB if the matrix $\bar{\boldsymbol{A}}$ is Hurwitz and the disturbance $\boldsymbol{d}(t)$ is bounded.

Behavior of the Continuous State-Feedback Loop. The control input generated by the state-feedback controller (5.4) is given by

$$\boldsymbol{u}(t) = -\boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}t}\boldsymbol{x}_0 - \int_0^t \boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-\tau)}\boldsymbol{E}\boldsymbol{d}(\tau)\,\mathrm{d}\tau.$$

This equation shows that the input $\boldsymbol{u}(t)$ does not only depend upon the initial state $\boldsymbol{x}_0$ but also on the disturbance input $\boldsymbol{d}(t)$. In the setting of event-based control, this aspect is important. If at time t_k the state $\boldsymbol{x}(t_k)$ is communicated to the control input generator, the control input generator is able to determine the same control input $\boldsymbol{u}(t_k) = -\boldsymbol{K}\boldsymbol{x}(t_k)$ as a continuous state-feedback controller. However, for all future times $t > t_k$, the control input generator has to know the disturbance $\boldsymbol{d}(t)$ for $t > t_k$:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}_{\mathrm{CT}}(t_k) - \int_{t_k}^t \boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-\tau)}\boldsymbol{E}\boldsymbol{d}(\tau)\,\mathrm{d}\tau, \quad t \geq t_k. \qquad (5.7)$$

This analysis shows two important facts:

- Continuous state-feedback control (5.4) gets the information about the current disturbance implicitly by the continuous communication of the current state $\boldsymbol{x}_{\mathrm{CT}}(t)$.
- Any feedback without continuous communication has to make assumptions about the disturbance to be attenuated. Unless the disturbance is measurable, any discontinuous feedback cannot have the same performance as the feedback loop with continuous communication.

The main idea of the event-based state-feedback approach discussed in the following is to replace the continuous state feedback (5.4) by an event-based controller so that the state $\boldsymbol{x}(t)$ of the event-based state-feedback loop remains, for all times t, in the neighborhood $\Omega_{\mathrm{e}}(\boldsymbol{x}_{\mathrm{CT}}(t))$ of the desired state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous state-feedback loop (5.5), (5.6).

5.2.3 Event-Based State Feedback

Control Input Generator. A direct consequence of the analysis in the preceding section is the fact that for the time $t \geq t_k$ the plant (5.1), (5.2) with the control input (5.7) behaves exactly like the continuous control loop (5.5), (5.6). If the control input generator uses Eq. (5.7) to determine the control input for $t \geq t_k$, then the best possible performance is obtained. To enable the control input generator to use this equation the state $\boldsymbol{x}(t_k)$ has to be measured and communicated to the control input generator, and an assumption concerning the disturbance has to be made.

In the following, the control input generator assumes that the disturbance is constant

$$\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k \quad \text{for} \quad t \geq t_k$$

with known magnitude $\hat{\boldsymbol{d}}_k$. Hence, it uses the equation

$$\boldsymbol{u}(t) = -\boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}(t_k) - \boldsymbol{K}\bar{\boldsymbol{A}}^{-1}\left(\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)} - \boldsymbol{I}_n\right)\boldsymbol{E}\hat{\boldsymbol{d}}_k, \quad t \geq t_k \tag{5.8}$$

which directly follows from Eq. (5.7) for constant disturbances, until it gets the next information $\boldsymbol{x}(t_{k+1})$. $\boldsymbol{I}_n$ denotes the identity matrix of size n.

> The **control input generator** determines the input (5.8) by means of a model of the continuous closed-loop system (5.5)
>
> $$\dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_{\mathrm{s}}(t) + \boldsymbol{E}\hat{\boldsymbol{d}}_k, \quad \boldsymbol{x}_{\mathrm{s}}(t_k^+) = \boldsymbol{x}(t_k),\ t \geq t_k \tag{5.9}$$
>
> $$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t). \tag{5.10}$$

Here, $\boldsymbol{x}_{\mathrm{s}}$ is used to denote *the state of the control input generator*. Note that the signal $\boldsymbol{u}(t)$ obtained by Eq. (5.8) is the same as the solution of (5.9), (5.10).

The time t_k^+ indicates the update of the model state $\boldsymbol{x}_{\mathrm{s}}$ with the measured state $\boldsymbol{x}(t_k)$, which the control input generator gets from the event generator at event time t_k. Figure 5.3 shows the block diagram of the control input generator. Suitable ways for determining the event time t_k and the disturbance estimate are presented later in this section.

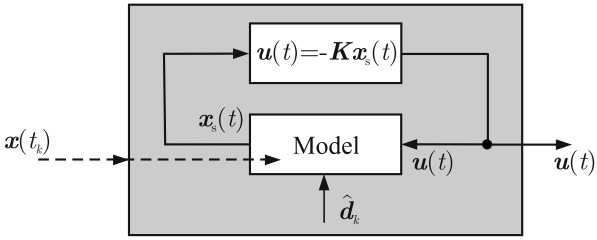

Fig. 5.3 Control input generator

Behavior of the Event-Based State-Feedback Loop. The analysis in this paragraph is valid for arbitrary event generators and arbitrary methods to estimate the disturbance magnitude $\hat{\boldsymbol{d}}_k$. It investigates the behavior of the event-based control loop in the time interval $[t_k, t_{k+1})$ between the consecutive event times t_k and t_{k+1}.

The plant (5.1), (5.2) together with the control input generator (5.9), (5.10) is described for the time period $[t_k, t_{k+1})$ by the state-space model

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_{\rm s}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{BK} \\ \boldsymbol{O} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t) + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{E} \end{pmatrix} \hat{\boldsymbol{d}}_k$$
$$\begin{pmatrix} \boldsymbol{x}(t_k^+) \\ \boldsymbol{x}_{\rm s}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}(t_k) \\ \boldsymbol{x}(t_k) \end{pmatrix}$$
$$\boldsymbol{y}(t) = (\boldsymbol{C} \quad \boldsymbol{O}) \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix}.$$

This model takes into account that the closed-loop system is subject to the disturbance $\boldsymbol{d}(t)$, whereas the control input generator uses the constant disturbance estimate $\hat{\boldsymbol{d}}_k$. The expression $\boldsymbol{x}_i(t_k^+) = \boldsymbol{x}_i(t_k)$ is used in the following to explicitly indicate that the respective state is not changed at the corresponding time instance.

By introducing the state transformation

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{O} & \boldsymbol{I}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix} \tag{5.11}$$

the following result can be obtained

Lemma 5.1. [248] *The output of the event-based state-feedback loop* (5.1), (5.2), (5.9), (5.10) *subject to the disturbance* $\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k + \boldsymbol{d}_\Delta(t)$ *consists of two components* $\boldsymbol{y}(t) = \boldsymbol{y}_{\rm s}(t) + \boldsymbol{y}_\Delta(t)$ *given by*

$$\boldsymbol{y}_{\rm s}(t) = \boldsymbol{C}{\rm e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}(t_k) + \boldsymbol{C}\bar{\boldsymbol{A}}^{-1}\left({\rm e}^{\bar{\boldsymbol{A}}(t-t_k)} - \boldsymbol{I}_n\right)\boldsymbol{E}\hat{\boldsymbol{d}}_k \tag{5.12}$$
$$\boldsymbol{y}_\Delta(t) = \int_{t_k}^{t} \boldsymbol{C}{\rm e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\boldsymbol{d}_\Delta(\tau){\rm d}\tau. \tag{5.13}$$

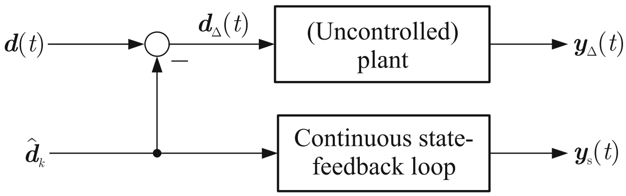

Fig. 5.4 Interpretation of Eqs. (5.12), (5.13)

The interpretation of this lemma is illustrated in Fig. 5.4. It shows three facts:

- The constant disturbance $\hat{\boldsymbol{d}}_k$ used by the control input generator, has the same effect on the event-based control as on the continuous state feedback. In the time interval $[t_k, t_{k+1})$ with the initial state $\boldsymbol{x}(t_k)$ both systems generate the output $\boldsymbol{y}_{\rm s}(t)$.
- The difference $\boldsymbol{d}_\Delta(t) = \boldsymbol{d}(t) - \hat{\boldsymbol{d}}_k$ between the actual disturbance $\boldsymbol{d}(t)$ and the constant disturbance estimate $\hat{\boldsymbol{d}}_k$ affects the (uncontrolled) plant and results in the output $\boldsymbol{y}_\Delta(t)$, which describes the difference between the outputs of the continuous state-feedback loop and the event-based state-feedback loop.
- For a good approximation $\hat{\boldsymbol{d}}_k$ of the disturbance $\boldsymbol{d}(t)$ in the time interval $[t_k, t_{k+1})$, i.e., $\boldsymbol{d}(t) - \hat{\boldsymbol{d}}_k \approx \boldsymbol{0}$, the plant subject to the open-loop control (5.8) behaves like the continuous state-feedback loop. No communication is necessary in this time interval.

According to the state transformation (5.11), the state $\boldsymbol{x}(t) = \boldsymbol{x}_{\rm s}(t) + \boldsymbol{x}_\Delta(t)$ of the event-based state-feedback loop can be decomposed into two components:

$$\boldsymbol{x}_{\rm s}(t) = {\rm e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}(t_k) + \bar{\boldsymbol{A}}^{-1}\left({\rm e}^{\bar{\boldsymbol{A}}(t-t_k)} - \boldsymbol{I}_n\right)\boldsymbol{E}\hat{\boldsymbol{d}}_k \tag{5.14}$$

$$\boldsymbol{x}_\Delta(t) = \int_{t_k}^{t} {\rm e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\boldsymbol{d}_\Delta(\tau){\rm d}\tau. \tag{5.15}$$

Like the output $\boldsymbol{y}_{\rm s}(t)$, the model state $\boldsymbol{x}_{\rm s}(t)$ is identical to the state trajectory of the continuous state-feedback system (5.5), (5.6) in the time interval $[t_k, t_{k+1})$ with initial state $\boldsymbol{x}_{\rm s}(t_k) = \boldsymbol{x}(t_k)$ and affected by the constant disturbance $\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k$.

Event Generator. Events are generated by comparing the measured state trajectory $\boldsymbol{x}(t)$ with the state trajectory $\boldsymbol{x}_{\rm s}(t)$ that would occur in the continuous state-feedback loop for the constant disturbance $\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k$. As the state $\boldsymbol{x}_{\rm s}(t)$ determined according to Eq. (5.9) represents the desired reference signal, the measured state $\boldsymbol{x}(t)$ should be kept in the surroundings

$$\Omega_{\rm s}(\boldsymbol{x}_{\rm s}(t)) = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_{\rm s}(t)\| \le \bar{e}\}$$

of this state with adjustable size $\bar{e}$.

The **event generator** triggers an event whenever the difference between the measured plant state $\boldsymbol{x}(t)$ and the reference state $\boldsymbol{x}_{\rm s}(t)$ reaches the event threshold $\bar{e}$:

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)\| = \bar{e}. \tag{5.16}$$

At this time instance t, which denotes the event time t_k, the state information $\boldsymbol{x}(t_k)$ is communicated to the control input generator.

In order to avoid a continuous transmission of the state $\boldsymbol{x}_{\rm s}(t)$ from the control input generator to the event generator, a copy of the control input generator is included in the event generator so that the event generator can determine the state $\boldsymbol{x}_{\rm s}(t)$ by means of Eq. (5.9).

As, at event time t_k, the state $\boldsymbol{x}_{\rm s}(t_k)$ is immediately updated with the measured state $\boldsymbol{x}(t_k)$, the following property holds.

Lemma 5.2. [248] *Event condition* (5.16) *ensures that the difference state* $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)$ *is bounded and remains in the set* Ω_Δ*:*

$$\boldsymbol{x}_\Delta(t) \in \Omega_\Delta = \{\boldsymbol{x}_\Delta \, : \, \|\boldsymbol{x}_\Delta\| \leq \bar{e}\}, \quad \forall t \geq 0.$$

Disturbance Estimator. The following investigations show how to get an estimate $\hat{\boldsymbol{d}}_k$ of the disturbance magnitude at the event time t_k. Assume that in the preceding time interval $[t_{k-1}, t_k)$ the disturbance estimate $\hat{\boldsymbol{d}}_{k-1}$ has been used. Consider now the difference $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)$ and assume that the disturbance $\boldsymbol{d}(t)$ has been constant in this time interval

$$\boldsymbol{d}(t) = \bar{\boldsymbol{d}} \quad \text{for} \quad t \in [t_{k-1}, t_k),$$

where $\bar{\boldsymbol{d}}$ is the actual disturbance magnitude, which usually differs from the estimate $\hat{\boldsymbol{d}}_k$. Equation (5.15) yields

$$\boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t) = \boldsymbol{A}^{-1}\left(\mathrm{e}^{\boldsymbol{A}(t - t_{k-1})} - \boldsymbol{I}_n\right)\boldsymbol{E}(\bar{\boldsymbol{d}} - \hat{\boldsymbol{d}}_{k-1})$$

which is used to determine, at time $t = t_k$, the unknown disturbance magnitude $\bar{\boldsymbol{d}}$:

The **disturbance estimator** determines the estimate $\hat{\boldsymbol{d}}_k$ recursively:

$$\hat{\boldsymbol{d}}_0 = \boldsymbol{0} \tag{5.17}$$

$$\hat{\boldsymbol{d}}_k = \hat{\boldsymbol{d}}_{k-1} + \left(\boldsymbol{A}^{-1}\left(\mathrm{e}^{\boldsymbol{A}(t_k - t_{k-1})} - \boldsymbol{I}_n\right)\boldsymbol{E}\right)^{+}(\boldsymbol{x}(t_k) - \boldsymbol{x}_{\rm s}(t_k)) \,. \tag{5.18}$$

The pseudoinverse $(\star)^+$ in Eq. (5.18) exists if, as usual, the number of disturbances is lower than the number of state variables ($n \geq l$) and the occurring matrices have full rank. Note that this disturbance estimation explicitly requires the existence of the inverse system matrix $\boldsymbol{A}^{-1}$ but it does not require the stability of the plant.

The disturbance estimator is included in the control input generator as well as in the event generator to provide both components with a current disturbance estimate at event times t_k, $(k = 0, 1, 2, ...)$. It has the following property.

Lemma 5.3. [248] *If at time t_1 the first event has been generated in the event-based state-feedback loop affected by a constant disturbance $\boldsymbol{d}(t) = \bar{\boldsymbol{d}}$, the disturbance estimator* (5.17), (5.18) *correctly determines the disturbance magnitude:*

$$\hat{\boldsymbol{d}}_1 = \bar{\boldsymbol{d}}.$$

If the disturbance $\boldsymbol{d}(t)$ changes, the estimate $\hat{\boldsymbol{d}}_k$ represents the "mean" value of $\boldsymbol{d}(t)$ in the time interval (t_0, t_1).

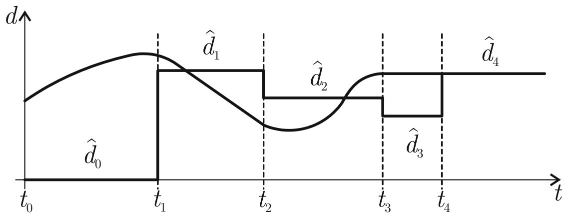

Fig. 5.5 Disturbance $d(t)$ and disturbance estimates $\hat{d}_0, \hat{d}_1, \hat{d}_2$...

For a scalar time-varying disturbance $d(t)$ the disturbance estimation is illustrated in Fig. 5.5, which shows the behavior of the disturbance $d(t)$ and the corresponding sequence of disturbance estimates. Here, $\hat{d}_1$ is the weighted average of the disturbance $d(t)$ for the time interval $[t_0, t_1)$. Similarly, $\hat{d}_2$ describes a weighted average of the actual disturbance for the time interval $[t_1, t_2)$. If the disturbance remains constant over two time intervals, then in the second time interval the estimate $\hat{d}_k$ coincides with the true magnitude of the disturbance. This happens in the example for $t \geq t_4$.

Summary of the Components. The event-based state-feedback loop has the structure depicted in Fig. 5.6. It has the following components:

- the plant (5.1), (5.2),
- the control input generator (5.9), (5.10) which also estimates the disturbance according to Eqs. (5.17), (5.18), and

- the event generator which includes a copy of the control input generator (5.9), (5.10) and the disturbance estimator (5.17), (5.18) and determines the event times t_k according to Eq. (5.16).

At event times t_k, $(k = 0, 1, 2, ...)$ the measured state information $\boldsymbol{x}(t_k)$ is sent from the event generator towards the control input generator and is used there as well as in the event generator to update the model state $\boldsymbol{x}_{\rm s}$ according to $\boldsymbol{x}_{\rm s}(t_k^+) = \boldsymbol{x}(t_k)$ and to determine the new disturbance estimate $\hat{\boldsymbol{d}}_k$. Since by assumption the data transmission is accomplished in no time, the models in the control input generator and the event generator work synchronously.

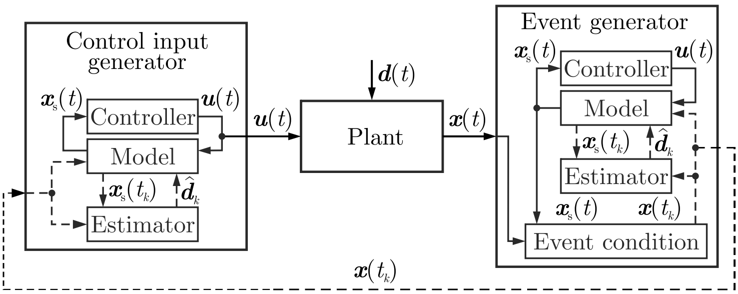

Fig. 5.6 Event-based state-feedback control loop

5.2.4 Main Properties of the Event-Based State-Feedback Loop

The central properties to be investigated when considering event-based control concern the stability and the communication over the feedback link. The main results of the subsequent analysis are the following:

- The state $\boldsymbol{x}(t)$ of the event-based control loop is *GUUB* and there exists an upper bound on its approximation error in terms of emulating the behavior of the continuous state-feedback loop (Theorem 5.1).
- There exists a lower bound on the minimum inter-event time (Theorem 5.2).
- If the disturbances are sufficiently small, no event is generated for $t > 0$ (Lemma 5.4).

Comparison between the Event-Based and the Continuous State-Feedback Loop. The following theorem compares the event-based control loop with the continuous state-feedback loop.

Theorem 5.1. [248] *The difference*

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{CT}}(t)$$

between the state $\boldsymbol{x}(t)$ *of the event-based state-feedback loop* (5.1), (5.2), (5.9), (5.10), (5.16) – (5.18) *and the state* $\boldsymbol{x}_{\mathrm{CT}}(t)$ *of the continuous state-feedback loop* (5.5), (5.6) *is bounded from above by*

$$\|\boldsymbol{e}(t)\| \le e_{\max} = \bar{e} \cdot \int_0^\infty \left\| \mathrm{e}^{\bar{\boldsymbol{A}}\tau} \boldsymbol{B}\boldsymbol{K} \right\| \mathrm{d}\tau. \tag{5.19}$$

This theorem shows that the event-based controller can be made to mimic a continuous state feedback system with arbitrary precision by accordingly choosing the event threshold $\bar{e}$. It can be used to determine for every tolerable upper bound on the approximation error $\|\boldsymbol{e}(t)\|$ the event threshold $\bar{e}$. The price for a higher precision (smaller $e_{\max}$) is a more frequent communication between the event generator and the control input generator. The state $\boldsymbol{x}(t)$ remains in the set

$$\boldsymbol{x}(t) \in \Omega_{\mathrm{e}}(\boldsymbol{x}_{\mathrm{CT}}(t)) = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{CT}}(t)\| \le e_{\max}\},$$

which describes a bounded neighborhood of the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous state-feedback loop for all times t as depicted in Fig. 5.7. Hence, as the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous state-feedback loop is *GUUB*, the state $\boldsymbol{x}(t)$ of the event-based state-feedback loop is *GUUB* as well.

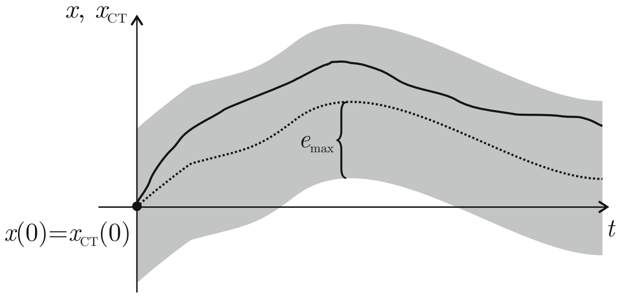

Fig. 5.7 Behavior of the event-based feedback loop: Solid line: state variable $x(t)$ of the event-based control loop; dotted line: state variable $x_{\mathrm{CT}}(t)$ of the continuous control loop

Minimum Inter-event Time. This paragraph shows that the minimum inter-event time

$$T_{\min} = \min_k (t_{k+1} - t_k)$$

of the event-based state-feedback loop is bounded from below and depends upon the disturbance $\boldsymbol{d}(t)$. Assume that the disturbance estimation error $\boldsymbol{d}_\Delta(t) = \boldsymbol{d}(t) - \hat{\boldsymbol{d}}_k$ is bounded by

$$\|\boldsymbol{d}_\Delta(t)\| \leq \gamma\, d_{\max} \quad \text{for} \quad t \geq 0, \tag{5.20}$$

with $0 \leq \gamma \leq 2$. The minimum inter-event time $T_{\min}$ is given by

$$T_{\min} = \arg\min_t \max_{\boldsymbol{d}_\Delta} \left\{ \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \boldsymbol{E} \boldsymbol{d}_\Delta(\tau)\, \mathrm{d}\tau \right\| = \bar{e} \right\}. \tag{5.21}$$

> **Theorem 5.2.** [248] *For any bounded disturbance* $\boldsymbol{d}(t)$*, the minimum inter-event time* $T_{\min}$ *of the event-based state-feedback loop* (5.1), (5.2), (5.9), (5.10), (5.16) – (5.18) *is bounded from below by* $\bar{T}$ *(*$T_{\min} \geq \bar{T}$*) given by*
>
> $$\bar{T} = \arg\min_t \left\{ \int_0^t \left\| \mathrm{e}^{\boldsymbol{A}\tau} \boldsymbol{E} \right\| \mathrm{d}\tau = \frac{\bar{e}}{\gamma\, d_{\max}} \right\}. \tag{5.22}$$

This theorem highlights how the communication depends on the disturbances. This phenomenon contrasts with sampled-data control, where the sampling frequency is chosen with respect to the plant properties (time constants) rather than the disturbance magnitude.

As the lower bound for the inter-event time decreases by decreasing the event threshold $\bar{e}$ and increases for large thresholds $\bar{e}$, Theorems 5.1 and 5.2 show that a higher precision generally leads to a more frequent communication from the event generator towards the control input generator.

Small Disturbances. The disturbance is represented in the following as $\boldsymbol{d}(t) = \bar{d}\tilde{\boldsymbol{d}}(t)$, where $\tilde{\boldsymbol{d}}(t)$ is an arbitrary finite vector function satisfying the inequality

$$\|\tilde{\boldsymbol{d}}(t)\| \leq 1 \quad \text{for} \quad t \geq 0$$

and $\bar{d}$ is the disturbance magnitude.

Lemma 5.4. [248] *Suppose that the plant* (5.1), (5.2) *is asymptotically stable. Then, for every bounded disturbance* $\boldsymbol{d}(t) = \bar{d}\tilde{\boldsymbol{d}}(t)$ *with magnitude* $\bar{d}$ *satisfying the relation*

$$|\bar{d}| < \frac{\bar{e}}{\int_0^\infty \|\mathrm{e}^{\boldsymbol{A}\tau} \boldsymbol{E}\|\, \mathrm{d}\tau} = \bar{d}_{\mathrm{UD}} \tag{5.23}$$

the event generator does not generate any event for $t > 0$.

This result gives a quantitative bound for the disturbance, for which no feedback occurs after the initial event at time $t = 0$. It shows that in event-based control the communication is adapted to the severity of the disturbance. If the disturbance is small enough, no feedback is necessary at all to meet the performance requirements.

5.2.5 Extensions

The state feedback approach to event-based control presented in this section has been extended in various ways. The main purpose of these extensions lies in the relaxation of the assumptions stated in Section 5.2.1.

In order to make the approach more attractive for practical applications, an *event-based PI controller* has been developed [222]. The extended scheme guarantees setpoint tracking for constant reference and disturbance signals while significantly reducing the communication compared to sampled-data PI control.

The effect of *model uncertainties* has been analyzed by specifying upper bounds on the uncertainties of the model parameters. The analysis shows that model uncertainties affect both the approximation accuracy and the frequency of communication but can be compensated by using more involved disturbance estimators [219].

For dealing with immeasurable state variables, an *event-based output-feedback* control has been proposed in [224]. There, a state observer is included in the event generator in order to determine an approximate state $\hat{\boldsymbol{x}}(t)$ of the plant state $\boldsymbol{x}(t)$ based on the measured output $\boldsymbol{y}(t)$. By using an adapted event condition which monitors the difference between the observer state $\hat{\boldsymbol{x}}(t)$ and the model state $\boldsymbol{x}_{\mathrm{s}}(t)$ and by sending the observer state $\hat{\boldsymbol{x}}(t_k)$ at event times to the control input generator, a bound on the approximation error and a minimum inter-event time can be guaranteed.

In networked control systems, the assumption of having an ideal communication channel is often violated. Hence, non-ideal effects like *transmission delays, packet loss or a quantization* of the transmitted information have to be taken into account. By modifying the structure of the event-based control loop mainly in terms of adapting the update mechanism, a stable behavior and a bounded communication can be preserved in all three cases if certain conditions on the delay, the number of consecutive packet losses or the resolution of the quantization are met [221, 223, 225].

In this section, the event-based state-feedback loop has been analyzed by evaluating the system behavior in between consecutive events. A different approach to the analysis has been presented in [351], based on the formulation of the event-based state-feedback loop as an impulsive system and which allows to uniformly investigate the system dynamics at and between event times. In contrast to the analysis presented in this section which only proves ultimate boundedness, the method presented in [351] also detects the asymptotic stability of the undisturbed event-based control system with stable plant dynamics.

The scheme presented in this section refers to a continuous realization of its components. As the control input generator and the event generator need to be implemented on digital hardware, a discrete-time approach becomes important. The main difference to the continuous approach is given by the fact that events cannot be triggered at any time but only at the sampling instants. The results obtained in [158, 220] are very similar to the continuous situation but, as expected, deteriorate with an increasing sampling period.

Finally, to relax the necessity of linear dynamics, an extension to nonlinear systems which are input-output linearizable has been investigated. Basically, the consideration of nonlinear dynamics requires a more involved event condition by means of which the desired properties of event-based control in terms of its stability and the boundedness of the communication can be proven [348, 349].

Example 5.1 *Event-based control of a thermofluid process*

This example illustrates the behavior of the event-based state-feedback loop in different scenarios. The plant is the thermofluid process shown in Fig. 5.8. The level $x_1(t)$ and the temperature $x_2(t)$ of the liquid in the tank TB have to be stabilized at the set-points by using the inflow $u_1(t)$ from tank T3 and the heating power $u_2(t)$ as control inputs. Hot water inflow from tank HW is the scalar disturbance $d(t)$ to be attainuated.

The linearized model of the plant is given by

$$\dot{\boldsymbol{x}}(t) = 10^{-3}\begin{pmatrix} -0.8 & 0 \\ -1\cdot 10^{-7} & -1.7 \end{pmatrix}\boldsymbol{x}(t) + 10^{-3}\begin{pmatrix} 211 & 0 \\ -108 & 20 \end{pmatrix}\boldsymbol{u}(t) + 10^{-3}\begin{pmatrix} 148 \\ -80 \end{pmatrix} d(t)$$

with $\boldsymbol{x}_0 = \boldsymbol{0}$. The controller is chosen to be

$$\boldsymbol{K} = \begin{pmatrix} 0.08 & -0.02 \\ 0.17 & 0.72 \end{pmatrix},$$

for which the continuous closed-loop system (5.5), (5.6) is stable and has desired disturbance attenuation properties. The event threshold $\bar{e}$ is set to $\bar{e} = 2$ and the event generator uses the supremum norm, for which the event condition reads as

$$\|\boldsymbol{x}_\Delta(t)\|_\infty = \|\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\|_\infty = 2. \tag{5.24}$$

Simulation Results. In the first investigation (left-hand side of Fig. 5.9) the plant is subject to a constant disturbance $d(t) = \bar{d}$ drawn by the solid line in the top subplot. After the initializing event at time $t_0 = 0$, an event takes place at time t_1 due to the level behavior, where the equality

$$|x_{\Delta,1}(t_1)| = |x_1(t_1) - x_{\mathrm{s},1}(t_1)| = 2 \tag{5.25}$$

jan.lunze@ruhr-uni-bochum.de