Jan Lunze

Linear Output Regulation
- A Summary -

$$\boldsymbol{A\Pi} + \boldsymbol{P} + \boldsymbol{B\Gamma} = \boldsymbol{\Pi A}_{\mathrm{v}}$$
$$\boldsymbol{Q} + \boldsymbol{C\Pi} = \boldsymbol{0}$$

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Abstract. The report explains the well-known regulator equations of linear multivariable control and demonstrates their application by an example. In the linear output regulation problem, an exosystem generates an input to the plant, which can be considered as a command signal or a disturbance. The aim is to find a controller that makes the control error to vanish asymptotically. Then the system is said to be regulated or to track the command signal. Both situations are summarised under the term of output regulation.

Contents

Summary of the regulator problem and its solution

Consider the plant

$$\Sigma_{\rm S}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{P}\boldsymbol{x}_{\rm v}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{e}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{Q}\boldsymbol{x}_{\rm v}(t) \end{array} \right. \tag{1}$$

with the m-dimensional input $\boldsymbol{u}(t)$ and the p-dimensional output $\boldsymbol{e}(t)$ that is interpreted as the control error. The $n_{\rm v}$–dimensional signal $\boldsymbol{x}_{\rm v}(t)$ is generated by the exosystem

$$\Sigma_{\rm v}: \quad \dot{\boldsymbol{x}}_{\rm v}(t) = \boldsymbol{A}_{\rm v}\boldsymbol{x}_{\rm v}(t), \quad \boldsymbol{x}_{\rm v}(0) = \boldsymbol{x}_{\rm v0} \tag{2}$$

(Fig. 1). The **regulator problem** is to find a controller such that the following three requirements are satisfied:

- *Asymptotic stability:* The controlled plant subject to the input $\boldsymbol{x}_{\rm v}(t) = \boldsymbol{0}$ should be asymptotically stable.
- *Asymptotic regulation:* The control error $\boldsymbol{e}(t)$ should vanish asymptotically for all initial states $\boldsymbol{x}_0$ and $\boldsymbol{x}_{\rm v0}$ of the plant $\Sigma_{\rm S}$ and of the exosystem $\Sigma_{\rm v}$:
$$\lim_{t\to\infty} \|\boldsymbol{e}(t)\| = 0. \tag{3}$$
- *Robustness:* The requirement (ii) should be satisfied for all plant parameter perturbations that do not de-stabilise the controlled plant.

The following **assumptions** are made:

(A1) The matrix $\boldsymbol{A}_{\rm v}$ of the exosystem $\Sigma_{\rm v}$ has only eigenvalues with nonnegative real part.

(A2) The pair $(\boldsymbol{A}, \boldsymbol{B})$ of the plant $\Sigma_{\rm S}$ is stabilisable.

(A3) The pair $\left(\begin{pmatrix} \boldsymbol{A}_{\rm v} & \boldsymbol{O} \\ \boldsymbol{P} & \boldsymbol{A} \end{pmatrix}, (\boldsymbol{Q},\ \boldsymbol{C}) \right)$ is detectable.

The following **results** are reviewed in the succeeding sections. The regulator problem is well-posed and a solution exists if and only if the eigenvalues of $\boldsymbol{A}_{\rm v}$ are not invariant zeros of the plant:

$$\text{rank} \begin{pmatrix} \mu\boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{pmatrix} = n + p, \quad \mu \in \sigma(\boldsymbol{A}_{\rm v}). \tag{4}$$

A solution to the regulator problem is obtained as follows:

1. Solve the regulator equations for the matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$:
$$\text{Regulator equations:} \quad \left\{ \begin{array}{rcl} \boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{P} + \boldsymbol{B}\boldsymbol{\Gamma} & = & \boldsymbol{\Pi}\boldsymbol{A}_{\rm v} \\ \boldsymbol{Q} + \boldsymbol{C}\boldsymbol{\Pi} & = & \boldsymbol{O}. \end{array} \right. \tag{5}$$
2. Find a state-feedback controller $\boldsymbol{K}$ such that $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ is Hurwitz.
3. Design a state observer that generates, from the measured control error $\boldsymbol{e}(t)$, the approximate states $\hat{\boldsymbol{x}}_{\rm v}(t)$ and $\hat{\boldsymbol{x}}(t)$.
4. Then the following controller solves the regulator problem:
$$C: \quad \boldsymbol{u}(t) = -\boldsymbol{K}\hat{\boldsymbol{x}}(t) + \boldsymbol{L}\hat{\boldsymbol{x}}_{\rm v}(t) \tag{6}$$
with
$$\boldsymbol{L} = \boldsymbol{\Gamma} + \boldsymbol{K}\boldsymbol{\Pi}. \tag{7}$$

1 Introduction

The regulator problem concerns the situation depicted in Fig. 1. The exosystem $\Sigma_{\rm v}$ generates the input $\boldsymbol{x}_{\rm v}(t)$ to the plant $\Sigma_{\rm S}$. The *regulator problem* is to find a control input $\boldsymbol{u}(t)$ such that the output $\boldsymbol{e}(t)$, which represents the control error, vanishes asymptotically

$$\lim_{t\to\infty} \|\boldsymbol{e}(t)\| = 0$$

for all initial states $\boldsymbol{x}_0$ of the plant and $\boldsymbol{x}_{\rm v0}$ of the exosystem. Then one says that the *output is regulated* or *asymptotic regulation* occurs.

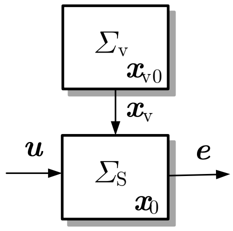

Fig. 1: Regulator problem

A solution to the regulator problem exists if and only if the *regulator equations* (5) given in the summary above have, for given parameters $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Q})$ of the plant $\Sigma_{\rm S}$ and $\boldsymbol{A}_{\rm v}$ of the exosystem $\Sigma_{\rm v}$, a solution for the unknown matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$. Although these equations are found in literature in several places, only a few explanations of them together with advices how to use them in order to find a controller that solves this problem have been published.

When looking at the equations (5), one may ask the following questions:

- What do the regulator equations say? Why are these equations necessary and sufficient to ensure asymptotic regulation?
- How can these equations be solved for the unknown matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$?
- Under what conditions does a solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ of these equations exist?
- How can the solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ of the regulator equations be used to design a feedback controller that ensures asymptotic regulation?

This report answers these questions for all who have the textbook knowledge provided by [12] and [13]. In contrast, the regulator equations have been explained in literature like in the monographs [9], [11] and [16] by using advanced topics of control theory like the McMillan normal form of transfer function matrices or the geometric approach to system theory.

In the original publications, the problem stated above has been posed as the *general servomechanism problem* by E. J. DAVISON around 1972 in [2, 3, 4] and as the *regulator problem* by B. A. FRANCIS around 1977 in [5]. In the given and several other references the general fact has been extensively discussed that a feedforward solution cannot be robust, but may solve the problem with the requirements (i) and (ii) included in the regulator problem below for exactly known plants, whereas a feedback solution may satisfy the additional robustness requirement (iv). The feedback that solves the regulator problem is called a *regulator*. Both solutions will be reviewed in this report.

The summary on p. v shows that the regulator theory provides two main results:

- **Existence of a regulator:** It is shown that under the assumptions (A1) – (A3) a solution to the regulator problem exists if and only if the condition (4) is satisfied, which means that the eigenvalues of $\boldsymbol{A}_{\rm v}$ must not be invariant zeros of the plant $\Sigma_{\rm S}$.
- **Design of a regulator:** A feedback controller solving the regulator problem can be found by using the solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ of the regulator equations (5) as follows. Design a state observer that reconstructs, by using the measured control error $\boldsymbol{e}(t)$, the state $\boldsymbol{x}_{\rm v}(t)$ of the exosystem and the state $\boldsymbol{x}(t)$ of the plant and denote these observation results by $\hat{\boldsymbol{x}}_{\rm v}(t)$ or $\hat{\boldsymbol{x}}(t)$, respectively. Furthermore, find a stabilising state feedback of the plant with controller matrix $\boldsymbol{K}$. Then a regulator is obtained by combining the state observer with the controller (6) that has the controller matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ given by eqn. (7).

2 Regulator problem

2.1 Problem statement

The regulator problem is usually formulated as follows. The external signal $\boldsymbol{x}_{\rm v}(t) \in {\rm I\!R}^{n_{\rm v}}$, which may be interpreted as a disturbance or a command signal (or both), is generated by the so-called *exosystem*

$$\Sigma_{\rm v}: \quad \dot{\boldsymbol{x}}_{\rm v}(t) = \boldsymbol{A}_{\rm v}\boldsymbol{x}_{\rm v}(t), \quad \boldsymbol{x}_{\rm v}(0) = \boldsymbol{x}_{\rm v0} \tag{8}$$

(Fig. 1). It is assumed that the matrix $\boldsymbol{A}_{\rm v}$ has only eigenvalues with nonnegative real part. The plant

$$\Sigma_{\rm S}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) &=& \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{P}\boldsymbol{x}_{\rm v}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{e}(t) &=& \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{Q}\boldsymbol{x}_{\rm v}(t) \end{array} \right. \tag{9}$$

has the m-dimensional input $\boldsymbol{u}(t)$ and the p-dimensional output $\boldsymbol{e}(t)$, which is interpreted as the control error that should be made to vanish asymptotically.

The *regulator problem* is to find a controller such that the following three requirements are satisfied:

(i) **Asymptotic stability:** The controlled plant subject to the input $\boldsymbol{x}_{\rm v}(t) = \boldsymbol{0}$ should be asymptotically stable.

(ii) **Asymptotic regulation:** The control error $\boldsymbol{e}(t)$ should vanish asymptotically for all initial states $\boldsymbol{x}_0$ and $\boldsymbol{x}_{\rm v0}$ of the plant $\Sigma_{\rm S}$ and of the exosystem $\Sigma_{\rm v}$:

$$\lim_{t\to\infty} \|\boldsymbol{e}(t)\| = 0. \tag{10}$$

(iv) **Robustness:** The requirement (ii) should be satisfied for all plant parameter perturbations that do not de-stabilise the controlled plant.

The requirements are enumerated as (i), (ii) and (iv) as in [12, 13], because control design problems usually have additional requirements on the transient behaviour of the control loop that are enumerated as (iii), which do not play any role in the regulator problem.

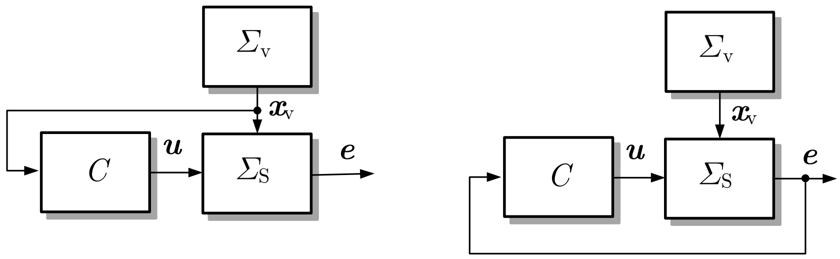

Fig. 2: Feedforward and feedback solution to the regulator problem

As the solution of the regulator problem, a feedforward controller or a feedback controller is used (Fig. 2). Both situations will be considered in the next sections.

2.2 Assumptions

The regulator problem is usually solved under the following assumptions:

(A1) The matrix $\boldsymbol{A}_{\rm v}$ of the exosystem $\Sigma_{\rm v}$ has only eigenvalues with nonnegative real part.

(A2) The pair $(\boldsymbol{A}, \boldsymbol{B})$ of the plant $\Sigma_{\rm S}$ is stabilisable.

(A3) The pair $\left(\begin{pmatrix} \boldsymbol{A}_{\rm v} & \boldsymbol{O} \\ \boldsymbol{P} & \boldsymbol{A} \end{pmatrix}, (\boldsymbol{Q}, \ \boldsymbol{C}) \right)$ is detectable.

If (A1) is violated, the exosystem $\Sigma_{\rm v}$ can be reduced without changing the regulator problem such that finally (A1) is satisfied. If $\boldsymbol{A}_{\rm v}$ does not have any eigenvalue with nonnegative real part, the regulator problem reduces to a stabilisation problem for the plant. If (A2) is violated, no solution to the regulator problem exists. The assumption (A3) requires that the series connection of $\Sigma_{\rm v}$ and $\Sigma_{\rm S}$ is detectable. This condition is necessary to find a state observer (52) for this series connection. In literature, this assumption is often replaced by the requirement that the pair $(\boldsymbol{A}, \boldsymbol{C})$ is detectable, but then the state $\boldsymbol{x}_{\rm v}(t)$ of the exosystem is assumed to be measurable.

The robustness requirement (iv) is important for the solution of the regulator problem, because it brings about a generality of the problem that removes specific parameter combinations, as the following investigations will show. The assumptions (A2) and (A3) satisfy these robustness requirements, because if the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilisable, then the pair $(\boldsymbol{A} + \delta\boldsymbol{A}, \boldsymbol{B} + \delta\boldsymbol{B})$ is also stabilisable for sufficiently small perturbations $\delta\boldsymbol{A}$ and $\delta\boldsymbol{B}$ and the same holds true for the detectability assumption.

2.3 Disturbance attenuation and command following

The regulator problem stated above includes the problems of disturbance attenuation and of asymptotic command following as special cases. In the left part of Fig. 3, $\Sigma_{\rm d}$ is a disturbance generator

$$\Sigma_{\rm d}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\rm d}(t) &=& \boldsymbol{A}_{\rm d}\boldsymbol{x}_{\rm d}(t), \quad \boldsymbol{x}_{\rm d}(0) = \boldsymbol{x}_{\rm d0} \\ \boldsymbol{d}(t) &=& \boldsymbol{C}_{\rm d}\boldsymbol{x}_{\rm d}(t). \end{array} \right.$$

The controller should compensate the disturbance $\boldsymbol{d}(t)$ acting on the plant

$$\Sigma_{\rm S}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) &=& \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &=& \boldsymbol{C}\boldsymbol{x}(t). \end{array} \right.$$

This problem can be formulated as a regulator problem by using the exosystem (8) with $\boldsymbol{x}_{\rm v}(t) = \boldsymbol{x}_{\rm d}(t)$,

$$\boldsymbol{A}_{\rm v} = \boldsymbol{A}_{\rm d} \quad \text{and} \quad \boldsymbol{x}_{\rm v0} = \boldsymbol{x}_{\rm d0}$$

and the plant model (9) with $\boldsymbol{e}(t) = \boldsymbol{y}(t)$,

$$\boldsymbol{P} = \boldsymbol{E}\boldsymbol{C}_{\rm d} \quad \text{and} \quad \boldsymbol{Q} = \boldsymbol{O}.$$

If the requirement (ii) is ensured, one says that the output $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$ is *regulated.*

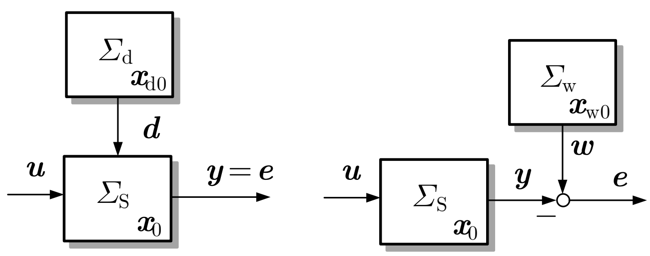

Fig. 3: Two interpretations of the regulator problem

The right part of Fig. 3 depicts the problem of command following with

$$\Sigma_{\rm w}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\rm w}(t) &=& \boldsymbol{A}_{\rm w}\boldsymbol{x}_{\rm w}(t), \quad \boldsymbol{x}_{\rm w}(0) = \boldsymbol{x}_{\rm w0} \\ \boldsymbol{w}(t) &=& \boldsymbol{C}_{\rm w}\boldsymbol{x}_{\rm w}(t) \end{array} \right.$$

representing the command signal generator. The output of the plant

$$\Sigma_{\rm S}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) &=& \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{e}(t) &=& -\tilde{\boldsymbol{C}}\boldsymbol{x}(t) + \boldsymbol{I}\boldsymbol{w}(t) \end{array} \right.$$

is the control error $\boldsymbol{e}(t)$. This problem is a regulator problem with the exosystem (8) with $\boldsymbol{x}_{\rm v}(t) = \boldsymbol{x}_{\rm w}(t)$,

$$\boldsymbol{A}_{\rm v} = \boldsymbol{A}_{\rm w} \quad \text{and} \quad \boldsymbol{x}_{\rm v0} = \boldsymbol{x}_{\rm w0}$$

and the plant (9) with

$$\boldsymbol{P} = \boldsymbol{O}, \quad \boldsymbol{C} = -\tilde{\boldsymbol{C}} \quad \text{and} \quad \boldsymbol{Q} = \boldsymbol{C}_{\rm w}.$$

If the requirement (ii) is satisfied, the output $\boldsymbol{y}(t) = -\tilde{\boldsymbol{C}}\boldsymbol{x}(t)$ is said to *track* the command signal $\boldsymbol{w}(t)$.

In the literature, both the disturbance attenuation problem and the tracking problem are said to pose the *servo problem* or the general regulator problem.

A comparison of Figs. 1 and 3 reveals that there is an important difference between the two interpretations considered in this section and the general regulator problem. In particular, as far as command tracking is concerned, the information that is usually available for a controller is the command signal $\boldsymbol{w}(t)$ and not the internal state $\boldsymbol{x}_{\rm w}(t)$ of the command signal generator $\Sigma_{\rm w}$. For the disturbance attenuation problem, the situation is "worse" in the sense that usually no information at all is available for a controller. Generally the disturbance $\boldsymbol{d}(t)$ is not measurable and the state $\boldsymbol{x}_{\rm d}(t)$ of the disturbance generator $\Sigma_{\rm d}$ cannot be assumed to be known. These aspects should be kept in mind when reading Section 4, where the state $\boldsymbol{x}_{\rm v}(t)$ of the exosystem is considered to be available as an input to the controller. The assumption of a measurable state $\boldsymbol{x}_{\rm v}(t)$ will be removed in Section 5, where the exosystem state is reconstructed from measured signals by a state observer.

2.4 Structure of the report

As a preliminary result, Section 3 considers a linear system subject to some input and shows how the stationary solution, which describes the system response for large time t, can be determined by a Sylvester equation. This equation will be extended in Section 4 to the regulator equations and it will be shown that the existence of a solution to these equations is a necessary and sufficient condition for a regulator to exist. The solution to the regulator equations can be used as a feedforward controller that solves the regulator problem with the requirements (i) and (ii). Section 5 extends this solution to a feedback controller. If the regulator equations are satisfied and a state observer is used to reconstruct the states of the exosystem and of the plant, the feedback controller given in this section solves the regulator problem with all the requirements (i), (ii) and (iv). Section 6 illustrates the results by an example.

3 Determination of the stationary solution of a linear system

3.1 Problem statement

This section shows that the stationary trajectory $\boldsymbol{x}_{\rm s}(t)$ or $\boldsymbol{y}_{\rm s}(t)$ of the state or the output, respectively, of a linear system

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) \end{array} \right. \tag{11}$$

can be determined by solving a Sylvester equation. The input to the n–th order system Σ is generated by the *input generator*

$$\Sigma_{\rm u} : \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\rm u}(t) & = & \boldsymbol{A}_{\rm u}\boldsymbol{x}_{\rm u}(t), \quad \boldsymbol{x}_{\rm u}(0) = \boldsymbol{x}_{\rm u0} \\ \boldsymbol{u}(t) & = & \boldsymbol{C}_{\rm u}\boldsymbol{x}_{\rm u}(t) \end{array} \right.$$

with the state $\boldsymbol{x}_{\rm u} \in {\rm I\!R}^{n_{\rm u}}$ (Fig. 4). It is assumed that

- $\boldsymbol{A}$ is Hurwitz,
- $\boldsymbol{A}_{\rm u}$ has only eigenvalues with nonnegative real part.

Whereas the first assumption implies that the free motion and the transient response of the system Σ vanish asymptotically, the second assumption has the consequence that neither $\boldsymbol{u}(t)$ nor $\boldsymbol{y}(t)$ and $\boldsymbol{x}(t)$ vanish asymptotically for almost all initial states $\boldsymbol{x}_{\rm u0}$ of the input generator $\Sigma_{\rm u}$. Due to these assumptions, the systems Σ and $\Sigma_{\rm u}$ do not have eigenvalues in common

$$\sigma(\boldsymbol{A}) \cap \sigma(\boldsymbol{A}_{\rm u}) = \emptyset, \quad \text{(no-resonance condition)}, \tag{12}$$

where $\sigma(.)$ denotes the spectrum of a square matrix.

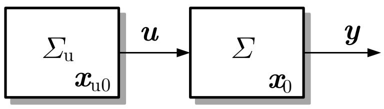

Fig. 4: Structure considered to determine the stationary solution of the system output $\boldsymbol{y}(t)$

Problem statement. The long-term behaviour of the system Σ is denoted by $\boldsymbol{x}_{\rm s}(t)$ and $\boldsymbol{y}_{\rm s}(t)$:

$$\lim_{t\to\infty} \|\boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)\| = 0 \tag{13}$$

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}_{\rm s}(t)\| = 0. \tag{14}$$

The question to be answered is: What is the relation between the trajectories $\boldsymbol{x}_{\rm s}(t)$ and $\boldsymbol{y}_{\rm s}(t)$ of the system Σ and the state $\boldsymbol{x}_{\rm u}(t)$ of the input generator $\Sigma_{\rm u}$?

3.2 Representation of $\boldsymbol{x}_{\rm s}(t)$ in terms of $\boldsymbol{x}_{\rm u}(t)$

The following investigations will show that there exists an $(n \times n_{\rm u})$–matrix $\boldsymbol{\Pi}$ such that the relation

$$\boldsymbol{x}_{\rm s}(t) = \boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) \tag{15}$$

holds. To determine $\boldsymbol{\Pi}$, introduce the state difference

$$\boldsymbol{e}_{\rm x}(t) = \boldsymbol{x}(t) - \boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t).$$

According to eqn. (13), the n–vector $\boldsymbol{e}_{\rm x}(t)$ vanishes asymptotically:

$$\lim_{\rm t\to\infty} \|\boldsymbol{e}_{\rm x}(t)\| = 0. \tag{16}$$

It follows the differential equation

$$\begin{aligned}
\dot{\boldsymbol{e}}_{\rm x}(t) &= \dot{\boldsymbol{x}}(t) - \boldsymbol{\Pi}\dot{\boldsymbol{x}}_{\rm u}(t) \\
&= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) - \boldsymbol{\Pi}\boldsymbol{A}_{\rm u}\boldsymbol{x}_{\rm u}(t) \\
&= \boldsymbol{A}\boldsymbol{e}_{\rm x}(t) + \boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) + \boldsymbol{B}\boldsymbol{C}_{\rm u}\boldsymbol{x}_{\rm u}(t) - \boldsymbol{\Pi}\boldsymbol{A}_{\rm u}\boldsymbol{x}_{\rm u}(t).
\end{aligned}$$

Since the matrix $\boldsymbol{A}$ is Hurwitz and the state $\boldsymbol{x}_{\rm u}(t)$ of the system $\Sigma_{\rm u}$ does not vanish, eqn. (16) implies that the following term on the right-hand side of the last equation is identical to zero

$$\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) + \boldsymbol{B}\boldsymbol{C}_{\rm u}\boldsymbol{x}_{\rm u}(t) - \boldsymbol{\Pi}\boldsymbol{A}_{\rm u}\boldsymbol{x}_{\rm u}(t) = \boldsymbol{0}, \quad t \geq 0$$

and, hence, that the relation

$$\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{B}\boldsymbol{C}_{\rm u} = \boldsymbol{\Pi}\boldsymbol{A}_{\rm u} \tag{17}$$

is satisfied. Equation (17) is a Sylvester equation[1], which has a unique solution $\boldsymbol{\Pi}$ for arbitrary matrices $\boldsymbol{B}\boldsymbol{C}_{\rm u}$ if and only if the spectra of the matrices $\boldsymbol{A}$ and $\boldsymbol{A}_{\rm u}$ are disjoint as stated in eqn. (12).

Lemma 1 *Under the assumptions stated above, the stationary state trajectory $\boldsymbol{x}_{\rm s}(t)$ of the system Σ has the representation (15), where the matrix $\boldsymbol{\Pi}$ is the solution of the Sylvester equation (17).*

Consequently, the state trajectory and the output trajectory of the system Σ have the properties (13) and (14) with

$$\begin{aligned} \boldsymbol{x}_{\rm s}(t) &= \boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) \quad (18)\\ \boldsymbol{y}_{\rm s}(t) &= \boldsymbol{C}\boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t), \quad (19) \end{aligned}$$

which can be represented by the $n_{\rm u}$–th order system

$$\Sigma_{\rm s}: \left\{ \begin{aligned} \dot{\boldsymbol{x}}_{\rm u}(t) &= \boldsymbol{A}_{\rm u}\boldsymbol{x}_{\rm u}(t), \quad \boldsymbol{x}_{\rm u}(0) = \boldsymbol{x}_{\rm u0}\\ \boldsymbol{x}_{\rm s}(t) &= \boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t)\\ \boldsymbol{y}_{\rm s}(t) &= \boldsymbol{C}\boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t). \end{aligned} \right.$$

3.3 Intuitive explanation of the result

The question arises why is it possible to get such a simple relation (15) between the states of the two systems Σ and $\Sigma_{\rm u}$ in the series connection of Fig. 4. An answer will be given in this section for matrices $\boldsymbol{A}$ and $\boldsymbol{A}_{\rm u}$ that are diagonalisable.

It is well known (e. g. from [13], S. 45) that the state $\boldsymbol{x}(t)$ and the output $\boldsymbol{y}(t)$ of the system (11) can be decomposed into three components

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}_{\rm free}(t) + \boldsymbol{x}_{\rm s}(t) + \boldsymbol{x}_{\rm t}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{y}_{\rm free}(t) + \boldsymbol{y}_{\rm s}(t) + \boldsymbol{y}_{\rm t}(t) \end{aligned}$$

with

$$\boldsymbol{x}_{\rm free}(t) = {\rm e}^{\boldsymbol{A}t}\boldsymbol{x}_0 \quad \text{and} \quad \boldsymbol{y}_{\rm free}(t) = \boldsymbol{C}{\rm e}^{\boldsymbol{A}t}\boldsymbol{x}_0$$

representing the free motion and $\boldsymbol{y}_{\rm s}(t)$ and $\boldsymbol{x}_{\rm s}(t)$ or $\boldsymbol{y}_{\rm t}(t)$ and $\boldsymbol{x}_{\rm t}(t)$ the stationary responses or the transient responses of Σ, respectively. To explain the difference between $\boldsymbol{x}_{\rm s}(t)$ and

[1] JAMES JOSEPH SYLVESTER (1814 – 1897), English mathematician

$\boldsymbol{x}_{\rm t}(t)$, denote the eigenvalues of $\boldsymbol{A}$ by λ_i, $(i = 1, 2, ..., n)$ and the eigenvalues of $\boldsymbol{A}_{\rm u}$ by μ_i, $(i = 1, 2, ..., n_{\rm u})$. The input to the system Σ generated by $\Sigma_{\rm u}$ is written as

$$\begin{aligned} \boldsymbol{u}(t) &= {\rm e}^{\boldsymbol{A}_{\rm u}t}\boldsymbol{x}_{\rm u0} \\ &= \sum_{i=1}^{n_{\rm u}} \boldsymbol{u}_i {\rm e}^{\mu_i t} x_{{\rm u}i} \qquad (20) \\ &= \boldsymbol{U} \begin{pmatrix} {\rm e}^{\mu_1 t} x_{\rm u1} \\ {\rm e}^{\mu_2 t} x_{\rm u2} \\ \vdots \\ {\rm e}^{\mu_{n_{\rm u}} t} x_{{\rm u}n_{\rm u}} \end{pmatrix} \end{aligned}$$

with the $(n_{\rm u} \times n_{\rm u})$–matrix

$$\boldsymbol{U} = (\boldsymbol{u}_1,\ \boldsymbol{u}_2, ..., \boldsymbol{u}_{n_{\rm u}})$$

of the eigenvectors of $\boldsymbol{A}_{\rm u}$ and the transformed initial state

$$\begin{pmatrix} x_{\rm u1} \\ x_{\rm u2} \\ \vdots \\ x_{{\rm u}n_{\rm u}} \end{pmatrix} = \boldsymbol{U}^{-1}\boldsymbol{x}_{\rm u0}.$$

The input consists of $n_{\rm u}$ exponential functions ${\rm e}^{\mu_i t}$, $(i = 1, 2, ..., n_{\rm u})$. Due to the no-resonance condition (12), the stationary state of the system Σ likewise consists of these exponential functions

$$\begin{aligned} \boldsymbol{x}_{\rm s}(t) &= \sum_{i=1}^{n_{\rm u}} \bar{\boldsymbol{x}}_i\, {\rm e}^{\mu_i t} x_{{\rm u}i} \qquad (21) \\ &= \overline{\boldsymbol{X}} \begin{pmatrix} {\rm e}^{\mu_1 t} x_{\rm u1} \\ {\rm e}^{\mu_2 t} x_{\rm u2} \\ \vdots \\ {\rm e}^{\mu_{n_{\rm u}} t} x_{{\rm u}n_{\rm u}} \end{pmatrix} \qquad (22) \end{aligned}$$

with appropriate n–dimensional vectors $\bar{\boldsymbol{x}}_i$, $(i = 1, 2, ..., n_{\rm u})$ that form the matrix

$$\overline{\boldsymbol{X}} = (\bar{\boldsymbol{x}}_1,\ \bar{\boldsymbol{x}}_2, ..., \bar{\boldsymbol{x}}_{n_{\rm u}})$$

(cf. [13], Abschn. 2.4.3). That is, the system Σ changes only the magnitude of the exponential functions ${\rm e}^{\mu_i t}$, but the exponents remain the same. In contrast, the transient behaviour $\boldsymbol{x}_{\rm t}(t)$ consists of n exponential functions ${\rm e}^{\lambda_i t}$, $(i = 1, 2, ..., n)$ with the exponents fixed by the eigenvalues of $\boldsymbol{A}$.

Therefore, it is reasonable to introduce a matrix $\boldsymbol{\Pi}$ that should satisfy the relation

$$\overline{\boldsymbol{X}} = \boldsymbol{\Pi}\boldsymbol{U}$$

or, equivalently,

$$\boldsymbol{\Pi} = \overline{\boldsymbol{X}}\boldsymbol{U}^{-1}.$$

The Sylvester equation (17) shows that this matrix $\boldsymbol{\Pi}$ exists and gives a way to determine $\boldsymbol{\Pi}$ without the knowledge of $\overline{\boldsymbol{X}}$.

3.4 Model transformation

The series connection $\Sigma \circ \Sigma_{\rm u}$ of the systems $\Sigma_{\rm u}$ and Σ is represented by

$$\Sigma \circ \Sigma_{\rm u}: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\boldsymbol{x}}_{\rm u}(t) \\ \dot{\boldsymbol{x}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm u} & \boldsymbol{O} \\ \boldsymbol{B}\boldsymbol{C}_{\rm u} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{x}(t) \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}_{\rm u}(0) \\ \boldsymbol{x}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{\rm u0} \\ \boldsymbol{x}_0 \end{pmatrix} \\ \boldsymbol{y}(t) & = & (\boldsymbol{O} \;\; \boldsymbol{C}) \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{x}(t) \end{pmatrix}. \end{array} \right. \tag{23}$$

This model can be simplified by using the solution $\boldsymbol{\Pi}$ obtained from the Sylvester equation (17) as follows. The state transformation

$$\begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{e}_{\rm x}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ -\boldsymbol{\Pi} & \boldsymbol{I} \end{pmatrix}}_{\boldsymbol{T}^{-1}} \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{x}(t) \end{pmatrix} \quad \text{with} \quad \boldsymbol{T} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{\Pi} & \boldsymbol{I} \end{pmatrix}$$

leads to the transformed model

$$\Sigma \circ \Sigma_{\rm u}: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\boldsymbol{x}}_{\rm u}(t) \\ \dot{\boldsymbol{e}}_{\rm x}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm u} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{e}_{\rm x}(t) \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}_{\rm u}(0) \\ \boldsymbol{e}_{\rm x}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{\rm u0} \\ \boldsymbol{x}_0 - \boldsymbol{\Pi}\boldsymbol{x}_{\rm u0} \end{pmatrix} \\ \boldsymbol{y}(t) & = & (\boldsymbol{C}\boldsymbol{\Pi} \;\; \boldsymbol{C}) \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{e}_{\rm x}(t) \end{pmatrix} \end{array} \right.$$

with a block-diagonal matrix that decouples the movement of $\boldsymbol{x}_{\rm u}(t)$ and $\boldsymbol{e}_{\rm x}(t)$. Obviously, eqns. (16), (18) and (19) hold and lead to the following statements:

$$\begin{array}{rcl} \boldsymbol{x}(t) \stackrel{t\to\infty}{\longrightarrow} \boldsymbol{x}_{\rm s}(t) & = & \boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) \\ & = & \boldsymbol{\Pi}\mathrm{e}^{\boldsymbol{A}_{\rm u}t}\boldsymbol{x}_{\rm u0} \\ \boldsymbol{y}(t) \stackrel{t\to\infty}{\longrightarrow} \boldsymbol{y}_{\rm s}(t) & = & \boldsymbol{C}\boldsymbol{\Pi}\boldsymbol{x}_{\rm u}(t) \\ & = & \boldsymbol{C}\boldsymbol{\Pi}\mathrm{e}^{\boldsymbol{A}_{\rm u}t}\boldsymbol{x}_{\rm u0}. \end{array}$$

3.5 Sylvester equation

There are several methods to solve the Sylvester equation (17)

$$-\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}_{\rm u} = \tilde{\boldsymbol{B}} \tag{24}$$

for given $(n_{\rm u} \times n_{\rm u})$–matrix $\boldsymbol{A}_{\rm u}$, $(n \times n)$–matrix $\boldsymbol{A}$ and $(n \times n_{\rm u})$–matrix $\tilde{\boldsymbol{B}} = \boldsymbol{B}\boldsymbol{C}_{\rm u}$ [14]. If the columns of the $(n \times n_{\rm u})$–matrix $\boldsymbol{\Pi}$ are put into a vector, linear equations for the unknown elements of this matrix appear as follows. Then by applying a linear-equation solver the matrix $\boldsymbol{\Pi}$ is obtained.

Vectorisation of the Sylvester equation. The results (A.1) and (A.2) on matrix vectorisation given in Appendix A lead to the following reformulations

$$\begin{aligned} -\mathrm{vec}(\boldsymbol{A}\boldsymbol{\Pi}) + \mathrm{vec}(\boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{u}}) &= \mathrm{vec}(\tilde{\boldsymbol{B}}) \\ -(\boldsymbol{I} \otimes \boldsymbol{A}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) + \left(\boldsymbol{A}_{\mathrm{u}}^{\mathrm{T}} \otimes \boldsymbol{I}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi}) &= \mathrm{vec}(\tilde{\boldsymbol{B}}) \\ \left(\boldsymbol{A}_{\mathrm{u}}^{\mathrm{T}} \otimes \boldsymbol{I} - \boldsymbol{I} \otimes \boldsymbol{A}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi}) &= \mathrm{vec}(\tilde{\boldsymbol{B}}) \end{aligned}$$

and to the linear equation

$$\boldsymbol{F}\boldsymbol{x} = \boldsymbol{b} \tag{25}$$

with

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{A}_{\mathrm{u}}^{\mathrm{T}} \otimes \boldsymbol{I}_n - \boldsymbol{I}_{n_{\mathrm{u}}} \otimes \boldsymbol{A} \\ \boldsymbol{x} &= \mathrm{vec}(\boldsymbol{\Pi}) \\ \boldsymbol{b} &= \mathrm{vec}(\tilde{\boldsymbol{B}}). \end{aligned}$$

For given $\boldsymbol{F}$ and $\boldsymbol{b}$ the vector $\boldsymbol{x}$ has to be found by a linear-equation solver.

In MATLAB, the function `sylvester(A, B, C)` is defined for the Sylvester equation

$$\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{B} = \boldsymbol{C}$$

and has to be invoked to solve eqn. (24) for the matrices `Au`, `A` and `Btilde` as follows:

```
» Pi = sylvester(-A, Au, Btilde);
```

Sylvester's theorem. Equation (25) and, equivalently, eqn. (24) should be universally solvable (i. e. solvable for arbitrary matrices $\tilde{\boldsymbol{B}}$). Then, according to Appendix B, the matrix $\boldsymbol{F}$ has to have full row rank:

$$\mathrm{rank}\,(\boldsymbol{F}) = n \cdot n_{\mathrm{u}}.$$

To understand the solvability condition given below, assume that the matrix $\boldsymbol{A}_{\mathrm{u}}$ is a diagonal matrix

$$\boldsymbol{A}_{\mathrm{u}} = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_{n_{\mathrm{u}}} \end{pmatrix}.$$

Then $\boldsymbol{F}$ is a block diagonal matrix with the blocks

$$\mu_i \boldsymbol{I}_n - \boldsymbol{A}, \quad i = 1, 2, ..., n_{\mathrm{u}}.$$

All these blocks have full rank if and only if the no-resonance condition (12) is satisfied. This result also holds true for non-diagonalisable matrices $\boldsymbol{A}_{\mathrm{u}}$.

Lemma 2 (Sylvester's theorem)
The Sylvester equation (24) is universally solvable if and only if the no-resonance condition (12) is satisfied.

From a system-theoretic viewpoint the existence condition stated by Lemma 2 is reasonable, because the state $\boldsymbol{x}_{\rm s}(t)$ of the plant $\Sigma_{\rm S}$ can be represented as a sum of the modes of the system $\Sigma_{\rm u}$ only if the eigenvalues of $\boldsymbol{A}_{\rm u}$ are not eigenvalues of $\boldsymbol{A}$. Otherwise, terms like $t^k{\rm e}^{\mu_i t}$ appear in $\boldsymbol{x}_{\rm s}(t)$, which cannot be represented by eqn. (18).

Example 1 *Analytical solution to the Sylvester equation*

To illustrate the solution of the Sylvester equation, consider a single-input single-output system (11)

$$\Sigma: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y(t) & = & \boldsymbol{c}^{\rm T}\boldsymbol{x}(t) \end{array} \right. \tag{26}$$

subject to the first-order input signal generator $\Sigma_{\rm u}$

$$\Sigma_{\rm u}: \left\{ \begin{array}{rcl} \dot{x}_{\rm u}(t) & = & \mu x_{\rm u}(t), \quad x_{\rm u}(0) = x_{\rm u0} \\ u(t) & = & x_{\rm u}(t), \end{array} \right.$$

where the vector $\boldsymbol{x}_{\rm u}(t)$ has been replaced by the scalar state $x_{\rm u}(t)$. Hence, the plant gets the input

$$u(t) = x_{\rm u0}\,{\rm e}^{\mu t}.$$

The Sylvester equation (17) reads as

$$\boldsymbol{A}\boldsymbol{\pi} + \boldsymbol{b} = \mu\boldsymbol{\pi}$$

with an unknown n–vector $\boldsymbol{\pi}$. This equation can be solved for $\boldsymbol{\pi}$ with the result

$$\boldsymbol{\pi} = (\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b},$$

because the inverse matrix exists since, by assumption, μ is not an eigenvalue of $\boldsymbol{A}$. Hence, the state and the output of the plant (26) have the following properties:

$$\begin{array}{l} \boldsymbol{x}(t) \stackrel{t\to\infty}{\longrightarrow} \boldsymbol{\pi}x_{\rm u}(t) = (\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}\,x_{\rm u0}\,{\rm e}^{\mu t} \\ y(t) \stackrel{t\to\infty}{\longrightarrow} \boldsymbol{c}^{\rm T}\boldsymbol{\pi}x_{\rm u}(t) = \boldsymbol{c}^{\rm T}(\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}\,x_{\rm u0}\,{\rm e}^{\mu t}. \end{array}$$

In the last line, the transfer function of (26) for the frequency $s = \mu$ appears as the "gain" with which the input is amplified by the plant:

$$G(\mu) = \boldsymbol{c}^{\rm T}(\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}. \quad \square$$

3.6 Summary

The stationary responses $\boldsymbol{x}_{\rm s}(t)$ and $\boldsymbol{y}_{\rm s}(t)$ of the system

$$\Sigma\circ\Sigma_{\rm u}: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\boldsymbol{x}}_{\rm u}(t) \\ \dot{\boldsymbol{x}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm u} & \boldsymbol{O} \\ \boldsymbol{B}\boldsymbol{C}_{\rm u} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{x}(t) \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}_{\rm u}(0) \\ \boldsymbol{x}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{\rm u0} \\ \boldsymbol{x}_0 \end{pmatrix} \\ \boldsymbol{y}(t) & = & (\,\boldsymbol{O} \;\; \boldsymbol{C}\,) \begin{pmatrix} \boldsymbol{x}_{\rm u}(t) \\ \boldsymbol{x}(t) \end{pmatrix} \end{array} \right. \tag{27}$$

can be determined by solving the Sylvester equation

$$\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{B}\boldsymbol{C}_{\mathrm{u}} = \boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{u}}. \tag{28}$$

The following statements hold:

- If the no-resonance condition

$$\sigma(\boldsymbol{A}) \cap \sigma(\boldsymbol{A}_{\mathrm{u}}) = \emptyset \tag{29}$$

  is satisfied, a unique solution $\boldsymbol{\Pi}$ to the Sylvester equation (28) exists.

- The no-resonance condition (29) is also necessary to make the Sylvester equation universally solvable for perturbed systems Σ_{S}.

- If the matrix $\boldsymbol{A}$ is Hurwitz, the state $\boldsymbol{x}(t)$ and the output $\boldsymbol{y}(t)$ of the system (27) approach the following stationary responses asymptotically:

$$\begin{aligned} \boldsymbol{x}(t) \overset{t\to\infty}{\longrightarrow} \boldsymbol{x}_{\mathrm{s}}(t) &= \boldsymbol{\Pi}\boldsymbol{x}_{\mathrm{u}}(t) \\ \boldsymbol{y}(t) \overset{t\to\infty}{\longrightarrow} \boldsymbol{y}_{\mathrm{s}}(t) &= \boldsymbol{C}\boldsymbol{\Pi}\boldsymbol{x}_{\mathrm{u}}(t). \end{aligned}$$

4 The regulator equations

4.1 Preliminary observations

The following analysis should give an intuitive explanation of how a solution to the regulator problem stated in Section 2.1 can look like. Assume that the plant Σ_{S} is asymptotically stable. Then, for arbitrary inputs, its free motion and its transient response vanish asymptotically and the output $\boldsymbol{e}(t)$ approaches the stationary solution $\boldsymbol{e}_{\mathrm{s}}(t)$

$$\boldsymbol{e}(t) \overset{t\to\infty}{\longrightarrow} \boldsymbol{e}_{\mathrm{s}}(t)$$

(for an explanation of the stationary solution cf. Section 3). To check the requirement (ii), only the stationary output $\boldsymbol{e}_{\mathrm{s}}(t)$ has to be investigated.

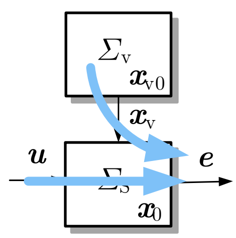

Fig. 5: Decomposition of the stationary control error $\boldsymbol{e}(t)$

As the plant (9) has the two inputs $\boldsymbol{x}_{\mathrm{v}}(t)$ and $\boldsymbol{u}(t)$, its stationary solution $\boldsymbol{e}_{\mathrm{s}}(t)$ has two components, each depending on one of the inputs:

$$\boldsymbol{e}_{\mathrm{s}}(t) = \boldsymbol{e}_{\mathrm{sx_v}}(t) + \boldsymbol{e}_{\mathrm{su}}(t)$$

(Fig. 5). As shown in Section 3, the component $\boldsymbol{e}_{\mathrm{sx_v}}(t)$ can be represented as

$$\boldsymbol{e}_{\mathrm{sx_v}}(t) = (\boldsymbol{C\Pi} + \boldsymbol{Q})\boldsymbol{x}_{\mathrm{v}}(t)$$

and it is a sum of the modes of the system Σ_{v}, all of which do not vanish asymptotically. Consequently, the requirement (ii) can only be satisfied if the signal $\boldsymbol{e}_{\mathrm{su}}(t)$ is

$$\boldsymbol{e}_{\mathrm{su}}(t) = -\boldsymbol{e}_{\mathrm{sx_v}}(t). \tag{30}$$

That is, the input $\boldsymbol{u}(t)$ has to consist of the same unstable modes as the exosystem state $\boldsymbol{x}_{\mathrm{v}}(t)$ and one may expect that some matrix $\boldsymbol{L}$ exists such that the solution to the regulator problem satisfies the relation

$$C: \quad \boldsymbol{u}(t) = \boldsymbol{L}\boldsymbol{x}_{\mathrm{v}}(t). \tag{31}$$

The regulator equations derived in the next section gives conditions on $\boldsymbol{L}$.

As the stationary solution $\boldsymbol{e}_{\mathrm{sx_v}}(t)$ consists of unstable modes, its compensation according to eqn. (30) requires that the input $\boldsymbol{u}(t)$ likewise consist of unstable modes. Hence, the requirement (ii) is satisfied if and only if the stationary solution $\boldsymbol{e}_{\mathrm{s}}(t)$ vanishes identically (and not only asymptotically):

$$\boldsymbol{e}_{\mathrm{s}}(t) = \boldsymbol{0}, \quad t \geq 0. \tag{32}$$

4.2 Derivation of the regulator equations

Consider the regulator problem with the requirements (i) and (ii), but without the robustness claim (iv). Assume that the plant Σ_{S} is asymptotically stable and, hence, due to the assumption (A1) the following condition is satisfied:

$$\sigma(\boldsymbol{A}) \cap \sigma(\boldsymbol{A}_{\mathrm{v}}) = \emptyset \quad \text{(no-resonance condition)}. \tag{33}$$

A solution (31) to the regulator problem exists under the conditions elaborated in the following analysis.

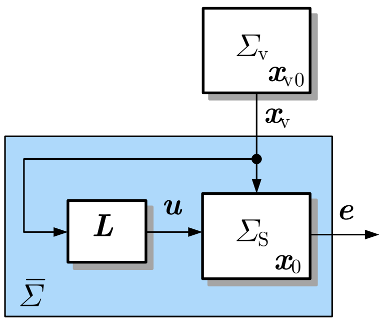

Fig. 6: Feedforward solution to the regulator problem

The overall system to be considered

$$\overline{\Sigma}\circ\Sigma_{\mathrm{v}}:\left\{\begin{array}{rcl}\begin{pmatrix}\dot{\boldsymbol{x}}_{\mathrm{v}}(t)\\ \dot{\boldsymbol{x}}(t)\end{pmatrix} & = & \left(\begin{array}{c:c}\boldsymbol{A}_{\mathrm{v}} & \boldsymbol{O}\\ \hdashline \boldsymbol{P}+\boldsymbol{B}\boldsymbol{L} & \boldsymbol{A}\end{array}\right)\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{x}(t)\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(0)\\ \boldsymbol{x}(0)\end{pmatrix}=\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}0}\\ \boldsymbol{x}_0\end{pmatrix}\\ \boldsymbol{e}(t) & = & (\,\boldsymbol{Q}\mid\boldsymbol{C}\,)\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{x}(t)\end{pmatrix}\end{array}\right. \tag{34}$$

is the series connection of the exosystem (8) and the controlled plant $\overline{\Sigma}$ described by eqns. (9), (31) (Fig. 6). It satisfies the requirement (i) due to the assumption that the matrix $\boldsymbol{A}$ is Hurwitz.

The stationary behaviour

$$\boldsymbol{x}_{\mathrm{s}}(t)=\boldsymbol{\Pi}\boldsymbol{x}_{\mathrm{v}}(t)$$

of the system Σ_{S} within the overall system $\overline{\Sigma}\circ\Sigma_{\mathrm{v}}$ can be determined by using the Sylvester equation

$$\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{P}+\boldsymbol{B}\boldsymbol{L}=\boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{v}}, \tag{35}$$

which is obtained for the system (34) in analogy to the Sylvester equation (28) for the system (27). Due to eqn. (33), the solution $\boldsymbol{\Pi}$ of eqn. (35) exists and is unique. Hence, the control error has the property

$$\boldsymbol{e}(t)\overset{t\to\infty}{\longrightarrow}\boldsymbol{e}_{\mathrm{s}}(t)=(\boldsymbol{Q}+\boldsymbol{C}\boldsymbol{\Pi})\boldsymbol{x}_{\mathrm{v}}(t).$$

The system (34) satisfies the requirement (ii) stated as eqn. (32) for arbitrary initial states $\boldsymbol{x}_{\mathrm{v}0}$ and $\boldsymbol{x}_0$ if and only if the relation

$$\boldsymbol{Q}+\boldsymbol{C}\boldsymbol{\Pi}=\boldsymbol{O} \tag{36}$$

holds. Hence, eqn. (31) provides a solution to the regulator problem if and only if the matrix $\boldsymbol{L}$ is chosen such that the solution $\boldsymbol{\Pi}$ of eqn. (35) satisfies the relation (36).

To illustrate this result, apply the state transformation

$$\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{e}_{\mathrm{x}}(t)\end{pmatrix}=\underbrace{\begin{pmatrix}\boldsymbol{I} & \boldsymbol{O}\\ -\boldsymbol{\Pi} & \boldsymbol{I}\end{pmatrix}}_{\boldsymbol{T}^{-1}}\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{x}(t)\end{pmatrix}\quad\text{with}\quad \boldsymbol{T}=\begin{pmatrix}\boldsymbol{I} & \boldsymbol{O}\\ \boldsymbol{\Pi} & \boldsymbol{I}\end{pmatrix}$$

to the model (34) to get, after using eqn. (35), the transformed model

$$\overline{\Sigma}\circ\Sigma_{\mathrm{v}}:\left\{\begin{array}{rcl}\begin{pmatrix}\dot{\boldsymbol{x}}_{\mathrm{v}}(t)\\ \dot{\boldsymbol{e}}_{\mathrm{x}}(t)\end{pmatrix} & = & \begin{pmatrix}\boldsymbol{A}_{\mathrm{v}} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{e}_{\mathrm{x}}(t)\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(0)\\ \boldsymbol{e}_{\mathrm{x}}(0)\end{pmatrix}=\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}0}\\ \boldsymbol{x}_0-\boldsymbol{\Pi}\boldsymbol{x}_{\mathrm{v}0}\end{pmatrix}\\ \boldsymbol{e}(t) & = & (\,\boldsymbol{Q}+\boldsymbol{C}\boldsymbol{\Pi}\quad\boldsymbol{C}\,)\begin{pmatrix}\boldsymbol{x}_{\mathrm{v}}(t)\\ \boldsymbol{e}_{\mathrm{x}}(t)\end{pmatrix}\end{array}\right.$$

with the solution

$$\boldsymbol{e}(t)=(\boldsymbol{Q}+\boldsymbol{C}\boldsymbol{\Pi})\mathrm{e}^{\boldsymbol{A}_{\mathrm{v}}t}\boldsymbol{x}_{\mathrm{v}0}+\boldsymbol{C}\mathrm{e}^{\boldsymbol{A}t}(\boldsymbol{x}_0-\boldsymbol{\Pi}\boldsymbol{x}_{\mathrm{v}0}).$$

Since $\boldsymbol{A}$ is Hurwitz, the second term vanishes asympotically. The control error meets the requirement (ii) if and only if the condition (36) is satisfied.

In summary, eqns. (35) and (36) are called the *regulator equations*:

$$\boldsymbol{A\Pi} + \boldsymbol{P} + \boldsymbol{BL} = \boldsymbol{\Pi A}_{\mathrm{v}} \tag{37}$$

$$\boldsymbol{Q} + \boldsymbol{C\Pi} = \boldsymbol{O}. \tag{38}$$

Lemma 3 (Feedforward solution of the regulator problem)
If the plant (9) is asymptotically stable, the regulator problem with the requirements (i) and (ii) is solved by the feedforward controller (31) if and only if there exists a matrix $\boldsymbol{\Pi}$ that satisfies the regulator equations (37), (38).

This lemma states a necessary and sufficient condition that the feedforward controller (31) solves the regulator problem. It is, in the stated version, a test for a given controller matrix $\boldsymbol{L}$, because for any fixed $\boldsymbol{L}$, the first regulator equation is a Sylvester equation that has a unique solution $\boldsymbol{\Pi}$ due to the assumption (33). With this solution, the second regulator equation has to be satisfied.

If the equations (37), (38) should also be used for design purposes, they have to be solved for given matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{A}_{\mathrm{v}}$ for the unknown matrices $\boldsymbol{\Pi}$ and $\boldsymbol{L}$. Section 4.4 will explain how this solution can be found. In specific cases like the following example an analytic solution is possible.

Example 2 *Feedforward solution to the regulator problem*

Consider the first-order exosystem

$$\Sigma_{\mathrm{v}}: \quad \dot{x}_{\mathrm{v}}(t) = \mu x_{\mathrm{v}}(t), \quad x_{\mathrm{v}}(0) = x_{\mathrm{v}0}$$

generating the command signal for the single-input single-output n–th order plant

$$\Sigma_{\mathrm{S}}: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), & \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ e(t) = -\tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}(t) + x_{\mathrm{v}}(t). \end{cases}$$

Assume that the matrix $\boldsymbol{A}$ is Hurwitz and the eigenvalue μ nonnegative. Command following of the controlled plant should be ensured by using the feedforward controller (31)

$$C: \quad u(t) = l x_{\mathrm{v}}(t)$$

where l is a scalar controller parameter.

To determine the parameter l, the regulator equations (37), (38) are written down as

$$\begin{aligned} \boldsymbol{A\pi} + \boldsymbol{b}l &= \boldsymbol{\pi}\mu \\ 1 - \tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{\pi} &= 0 \end{aligned}$$

with the n-dimensional vector $\boldsymbol{\pi}$. The first equation leads to the expression

$$\boldsymbol{\pi} = (\mu \boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}l$$

and the second equation to

$$l = \frac{1}{\tilde{\boldsymbol{c}}^{\mathrm{T}}(\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}}.$$

The inverse matrix exists due to the assumptions on $\boldsymbol{A}$ and μ. The result can be easily interpreted in terms of the transfer function

$$G(s) = -\tilde{\boldsymbol{c}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$$

of the plant between the input $u(t)$ and the output $e(t)$:

$$l = -\frac{1}{G(\mu)}.$$

To understand that this solution satisfies the requirement (ii), set-up the model of the controlled plant

$$\overline{\Sigma}: \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\, l x_{\mathrm{v}}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ e(t) &= -\tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}(t) + x_{\mathrm{v}}(t). \end{cases}$$

The transfer function of this system with input $x_{\mathrm{v}}(t)$ and output $e(t)$ is

$$\begin{aligned} \overline{G}(s) &= 1 - \tilde{\boldsymbol{c}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}\, l \\ &= 1 - \frac{\tilde{\boldsymbol{c}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}}{\tilde{\boldsymbol{c}}^{\mathrm{T}}(\mu\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}} \end{aligned}$$

with the property

$$\overline{G}(\mu) = 0. \tag{39}$$

As the free motion and the transient response of this system for the input $x_{\mathrm{v}}(t) = x_{\mathrm{v}0}\,\mathrm{e}^{\mu t}$ vanish asymptotically, the control error has the property

$$e(t) \overset{t\to\infty}{\longrightarrow} e_{\mathrm{s}}(t) = \overline{G}(\mu) x_{\mathrm{v}0}\,\mathrm{e}^{\mu t} = 0$$

as required.

Discussion. The last part of the example has shown that the controller (31) introduces a zero $s_0 = \mu$ into the system $\overline{\Sigma}$ consisting of the controller and the plant. This zero blocks the influence of the input $x_{\mathrm{v}}(t)$ towards the control error $e(t)$ as the output. Consequently, the plant output $y(t) = \tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}(t)$ asymptotically follows the command signal $x_{\mathrm{v}}(t)$. □

4.3 Extension to unstable plants

The results of Section 4.2 have been obtained under the condition that the plant is asymptotically stable. This section shows that the regulator equations can be easily extended for unstable plants.

Due to the assumption (A2), a state feedback with the controller matrix $\boldsymbol{K}$ exists that makes the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ Hurwitz to satisfy the requirement (i). The controller (31) is extended by this state feedback to get

$$C: \quad \boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) + \boldsymbol{L}\boldsymbol{x}_{\mathrm{v}}(t). \tag{40}$$

Then the model (34) of the overall system is replaced by

$$\overline{\Sigma} \circ \Sigma_{\rm v}: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\boldsymbol{x}}_{\rm v}(t) \\ \dot{\boldsymbol{x}}(t) \end{pmatrix} & = & \left(\begin{array}{c:c} \boldsymbol{A}_{\rm v} & \boldsymbol{O} \\ \hdashline \boldsymbol{P} + \boldsymbol{B}\boldsymbol{L} & \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{x}_{\rm v}(0) \\ \boldsymbol{x}(0) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{x}_{\rm v0} \\ \boldsymbol{x}_0 \end{pmatrix} \\ \boldsymbol{e}(t) & = & (\, \boldsymbol{Q} \mid \boldsymbol{C} \,) \begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \end{pmatrix}. \end{array} \right. \tag{41}$$

Now, the first regulator equation reads as

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{\Pi} + \boldsymbol{P} + \boldsymbol{B}\boldsymbol{L} = \boldsymbol{\Pi}\boldsymbol{A}_{\rm v}$$

and can be re-formulated to get the regulator equations in their final form

$$\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{P} + \boldsymbol{B}\boldsymbol{\Gamma} = \boldsymbol{\Pi}\boldsymbol{A}_{\rm v} \tag{42}$$

$$\boldsymbol{Q} + \boldsymbol{C}\boldsymbol{\Pi} = \boldsymbol{O} \tag{43}$$

with $\boldsymbol{\Gamma} = \boldsymbol{L} - \boldsymbol{K}\boldsymbol{\Pi}$. For any stabilising feedback matrix $\boldsymbol{K}$, the regulator equations (42), (43) are solved for $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ and the feedforward matrix is determined as $\boldsymbol{L} = \boldsymbol{\Gamma} + \boldsymbol{K}\boldsymbol{\Pi}$. The controller (40) is often re-formulated as

$$\boldsymbol{u}(t) = -\boldsymbol{K}(\boldsymbol{x}(t) - \boldsymbol{\Pi}\boldsymbol{x}_{\rm v}(t)) + \boldsymbol{\Gamma}\boldsymbol{x}_{\rm v}(t) \tag{44}$$

to show that the first part $-\boldsymbol{K}(\boldsymbol{x}(t) - \boldsymbol{\Pi}\boldsymbol{x}_{\rm v}(t))$ affects the plant behaviour only as long as the state $\boldsymbol{x}(t)$ is not equal to the stationary solution $\boldsymbol{x}_{\rm s}(t) = \boldsymbol{\Pi}\boldsymbol{x}_{\rm v}(t)$.

Theorem 1 (Feedforward solution of the regulator problem)
The regulator problem with the requirements (i) and (ii) is solved by the feedforward controller (44) if and only if there exists a solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ of the regulator equations (42), (43) and $\boldsymbol{K}$ makes the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ Hurwitz.

As the eqns. (37), (38) and (42), (43) have the same mathematical form, the regulator equations (42), (43) have a solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ if and only if eqns. (37), (38) have a solution $(\boldsymbol{\Pi}, \boldsymbol{L})$. That is, the introduction of the state feedback $\boldsymbol{K}$ does not change the existence conditions for a solution to the regulator problem.

Remark. The introduction of the feedback term $-\boldsymbol{K}\boldsymbol{x}(t)$ into the control law does not mean that the solution (40) to the regulator problem is a feedback controller with respect to $\boldsymbol{x}_{\rm v}(t)$, but the controller (40) remains to be a feedforward solution with respect to the requirement (ii). Hence, the robustness requirement (iv) is not satisfied.

4.4 Solution of the regulator equations

There are several numerical methods to solve linear matrix equations. To understand the way of solution, write the two regulator equations (42), (43) as

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Pi} \\ \boldsymbol{\Gamma} \end{pmatrix} \boldsymbol{A}_{\mathrm{v}} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Pi} \\ \boldsymbol{\Gamma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix}. \tag{45}$$

For given matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Q}$ and $\boldsymbol{A}_{\mathrm{v}}$, the two matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ should be found such that this equation is satisfied. Due to the multiplication of the left term by the matrix $\boldsymbol{A}_{\mathrm{v}}$ from the right, there is no explicit solution to this problem. One way to determine the required matrices, which will be explained below, uses a vectorisation of the equations that leads to a set of linear equations represented by

$$\boldsymbol{F}\boldsymbol{x} = \boldsymbol{b}, \tag{46}$$

which has to be solved, for a given matrix $\boldsymbol{F}$ and vector $\boldsymbol{b}$, by a linear-equation solver to get the unknown vector $\boldsymbol{x}$.

Reformulation of the regulator equations. With the preliminary transformations (A.1) and (A.2) derived in Appendix A, eqn. (45) or (42), (43) lead to the following equivalent statements

$$\begin{aligned}
\boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{v}} - \boldsymbol{A}\boldsymbol{\Pi} - \boldsymbol{B}\boldsymbol{\Gamma} &= \boldsymbol{P} \\
-\boldsymbol{C}\boldsymbol{\Pi} &= \boldsymbol{Q} \\
\mathrm{vec}\,(\boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{v}} - \boldsymbol{A}\boldsymbol{\Pi} - \boldsymbol{B}\boldsymbol{\Gamma}) &= \mathrm{vec}(\boldsymbol{P}) \\
\mathrm{vec}\,(\boldsymbol{\Pi}\boldsymbol{A}_{\mathrm{v}}) - \mathrm{vec}\,(\boldsymbol{A}\boldsymbol{\Pi}) - \mathrm{vec}\,(\boldsymbol{B}\boldsymbol{\Gamma}) &= \mathrm{vec}(\boldsymbol{P}) \\
\left(\boldsymbol{A}_{\mathrm{v}}^{\mathrm{T}} \otimes \boldsymbol{I}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi}) - (\boldsymbol{I} \otimes \boldsymbol{A}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) - (\boldsymbol{I} \otimes \boldsymbol{B}) \cdot \mathrm{vec}(\boldsymbol{\Gamma}) &= \mathrm{vec}(\boldsymbol{P}) \\
-\mathrm{vec}\,(\boldsymbol{C}\boldsymbol{\Pi}) &= \mathrm{vec}(\boldsymbol{Q}) \\
-(\boldsymbol{I} \otimes \boldsymbol{C}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) &= \mathrm{vec}(\boldsymbol{Q}) \\
\begin{pmatrix} \left(\boldsymbol{A}_{\mathrm{v}}^{\mathrm{T}} \otimes \boldsymbol{I}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi}) - (\boldsymbol{I} \otimes \boldsymbol{A}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) - (\boldsymbol{I} \otimes \boldsymbol{B}) \cdot \mathrm{vec}(\boldsymbol{\Gamma}) \\ -(\boldsymbol{I} \otimes \boldsymbol{C}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) \end{pmatrix} &= \begin{pmatrix} \mathrm{vec}(\boldsymbol{P}) \\ \mathrm{vec}(\boldsymbol{Q}) \end{pmatrix} \\
\begin{pmatrix} \left(\boldsymbol{A}_{\mathrm{v}}^{\mathrm{T}} \otimes \boldsymbol{I}\right) - (\boldsymbol{I} \otimes \boldsymbol{A}) & -(\boldsymbol{I} \otimes \boldsymbol{B}) \\ -(\boldsymbol{I} \otimes \boldsymbol{C}) & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\boldsymbol{\Pi}) \\ \mathrm{vec}(\boldsymbol{\Gamma}) \end{pmatrix} &= \begin{pmatrix} \mathrm{vec}(\boldsymbol{P}) \\ \mathrm{vec}(\boldsymbol{Q}) \end{pmatrix}
\end{aligned}$$

and to a linear equation of the form (46) with

$$\begin{aligned}
\boldsymbol{F} &= \begin{pmatrix} \left(\boldsymbol{A}_{\mathrm{v}}^{\mathrm{T}} \otimes \boldsymbol{I}\right) - (\boldsymbol{I} \otimes \boldsymbol{A}) & -(\boldsymbol{I} \otimes \boldsymbol{B}) \\ -(\boldsymbol{I} \otimes \boldsymbol{C}) & \boldsymbol{O} \end{pmatrix} \\
&= \boldsymbol{A}_{\mathrm{v}}^{\mathrm{T}} \otimes \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{O}_{n\times m} \\ \boldsymbol{O}_{n_{\mathrm{v}}\times n} & \boldsymbol{O}_{n_{\mathrm{v}}\times m} \end{pmatrix} - \boldsymbol{I}_{n_{\mathrm{v}}} \otimes \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O}_{p\times m} \end{pmatrix}
\end{aligned} \tag{47}$$

$$\boldsymbol{x} = \begin{pmatrix} \mathrm{vec}(\boldsymbol{\Pi}) \\ \mathrm{vec}(\boldsymbol{\Gamma}) \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} \mathrm{vec}(\boldsymbol{P}) \\ \mathrm{vec}(\boldsymbol{Q}) \end{pmatrix}. \tag{48}$$

After eqn. (46) has been solved for the unknown vector $\boldsymbol{x}$, the required matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ are obtained by reshaping the first $n \cdot n_{\rm v}$ elements of $\boldsymbol{x}$ as the $(n \times n_{\rm v})$–matrix $\boldsymbol{\Pi}$ and the remaining $m \cdot n_{\rm v}$ elements as the $(m \times n_{\rm v})$–matrix $\boldsymbol{\Gamma}$.

In a MATLAB script, these steps are carried out with the following commands. Store the required data in the matrices `A`, `B`, `C`, `P`, `Q` and `Av` and in the integer variables `n`, `m`, `p` and `nv`. Then the matrix `F` and the vector `b` are obtained according to eqns. (47) and (48) as follows:

```
» I_n = eye(n);
» I_nv =eye(nv);
» F=[kron(Av',I_n)+kron(I_nv,-A), kron(I_nv,-B);
   kron(I_n,-C) zeros(nv*p,nv*m))];
« b=[P(:); Q(:)];
```

The linear equation (46) is solved by the function call `x = F\b`, which finds the solution `x` in the least-square sense even if `x` is underdetermined or overdetermined by eqn. (46):

```
» x=F\b;
```

Then the required solutions are

```
» Pi = reshape(x(1:n*nv),n,nv);
» Gamma = reshape(x(n*nv+1:end),m,nv);
```

4.5 Solvability of the regulator equations

The statement of the regulator equations as eqn. (46) can be used to discuss the solvability of these equations. A linear equation (46) has a solution if and only if the $n_{\rm v}(n+p)$–vector $\boldsymbol{b}$ lies in the image of the $(n_{\rm v}(n+p) \times n_{\rm v}(n+m))$–matrix $\boldsymbol{F}$:

$$\text{rank}\,(\boldsymbol{F}) = \text{rank}\,(\boldsymbol{F}\ \boldsymbol{b}). \tag{49}$$

However, the robustness requirement (iv) can only be satisfied if the problem to find the vector $\boldsymbol{x}$ in eqn. (46) is well-posed as Appendix B discusses in detail. Equation (46) is universally solvable if and only if the matrix $\boldsymbol{F}$ has full row rank

$$\text{rank}\,(\boldsymbol{F}) = n_{\rm v}(n+p), \tag{50}$$

which is a stronger condition and ensures that any $n_{\rm v}(n+p)$–vector $\boldsymbol{b}$ satisfies eqn. (49). Equation (50) has two consequences. First, it can be satisfied only if $m \geq p$ holds, that is, if the plant $\Sigma_{\rm S}$ has at least as many inputs as outputs. Second, it leads to the condition (51) below on the plant zeros, which is obtained as follows.

Assume that the matrix $\boldsymbol{A}_{\rm v}$ is a diagonal matrix

$$\boldsymbol{A}_{\rm v} = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_{n_{\rm v}} \end{pmatrix}.$$

Then the matrix obtained from eqn. (47) is a block diagonal matrix

$$\boldsymbol{F} = \begin{pmatrix} \boldsymbol{F}_1 & & & \\ & \boldsymbol{F}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{F}_{n_\mathrm{v}} \end{pmatrix}$$

with the blocks

$$\boldsymbol{F}_i = \begin{pmatrix} \mu_i \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{pmatrix}, \quad i = 1, 2, ..., n_\mathrm{v}.$$

$\boldsymbol{F}$ has full row rank as required in eqn. (50) if and only if the relations

$$\mathrm{rank} \begin{pmatrix} \mu \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{pmatrix} = n + p, \quad \mu \in \sigma(\boldsymbol{A}_\mathrm{v}) \tag{51}$$

hold, where p is the dimension of $\boldsymbol{e}(t)$. The minus sign before $\boldsymbol{B}$ has been added to show the coincidence of this condition with the definition of the transmission zeros of the plant ([13], Definition 2.3). It has been shown in literature that the condition (51) is also obtained for non-diagonalisable matrices $\boldsymbol{A}_\mathrm{v}$.

Lemma 4 (Existence of a solution to the regulator equations)
The regulator equations (42), (43) are well-posed and a solution exists if and only if the condition (51) is satisfied, i. e., if the eigenvalues of $\boldsymbol{A}_\mathrm{v}$ are not transmission zeros of the plant Σ_S.

5 A robust feedback solution

5.1 State feedback solution to the regulator problem

The robustness requirement (iv) with respect to plant parameter variations can only be ensured by introducing a feedback C of the control error $\boldsymbol{e}(t)$ as shown in Fig. 7. This section will demonstrate that a combination of a Luenberger observer reconstructing the exosystem state $\boldsymbol{x}_\mathrm{v}(t)$ and the plant state $\boldsymbol{x}(t)$ with the feedforward control (40) leads to a feedback solution of the regulator problem that satisfies all the requirements (i), (ii) and (iv).

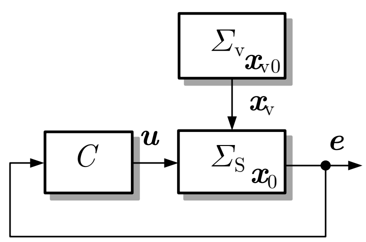

Fig. 7: Feedback solution of the regulator problem

Luenberger observer. As the state $\boldsymbol{x}_{\rm v}(t)$ of the system $\Sigma_{\rm v}$ and the state $\boldsymbol{x}(t)$ of the plant are generally not measurable, a Luenberger observer

$$O: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\hat{\boldsymbol{x}}}_{\rm v}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm v} & \boldsymbol{O} \\ \boldsymbol{P} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{B} \end{pmatrix} \boldsymbol{u}(t) \\ & & + \begin{pmatrix} \boldsymbol{G}_1 \\ \boldsymbol{G}_2 \end{pmatrix} (\boldsymbol{e}(t) - \hat{\boldsymbol{e}}(t)), \quad \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(0) \\ \hat{\boldsymbol{x}}(0) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v0} \\ \hat{\boldsymbol{x}}_0 \end{pmatrix} \\ \hat{\boldsymbol{e}}(t) & = & (\, \boldsymbol{Q} \;\; \boldsymbol{C} \,) \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \end{array} \right.$$

is used to reconstruct both state vectors. By inserting the observer feedback, the following representation is obtained:

$$O: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\hat{\boldsymbol{x}}}_{\rm v}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm v} - \boldsymbol{G}_1\boldsymbol{Q} & -\boldsymbol{G}_1\boldsymbol{C} \\ \boldsymbol{P} - \boldsymbol{G}_2\boldsymbol{Q} & \boldsymbol{A} - \boldsymbol{G}_2\boldsymbol{C} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{B} \end{pmatrix} \boldsymbol{u}(t) \\ & & + \begin{pmatrix} \boldsymbol{G}_1 \\ \boldsymbol{G}_2 \end{pmatrix} \boldsymbol{e}(t), \quad \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(0) \\ \hat{\boldsymbol{x}}(0) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v0} \\ \hat{\boldsymbol{x}}_0 \end{pmatrix} \\ \hat{\boldsymbol{e}}(t) & = & (\, \boldsymbol{Q} \;\; \boldsymbol{C} \,) \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix}. \end{array} \right. \tag{52}$$

The matrix $\boldsymbol{G} = (\boldsymbol{G}_1^{\rm T},\ \boldsymbol{G}_2^{\rm T})^{\rm T}$ has to be chosen so that the observer O is asymptotically stable, which can be ensured by using well-known state-feedback methods [13].

Feedback solution to the regulator problem. By using the reconstructed states $\hat{\boldsymbol{x}}_{\rm v}(t)$ and $\hat{\boldsymbol{x}}(t)$ the feedforward control law (40) can be implemented as follows:

$$\boldsymbol{u}(t) = (\boldsymbol{L} \quad - \boldsymbol{K}) \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix}. \tag{53}$$

Remember that the feedback matrix $\boldsymbol{K}$ makes the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ Hurwitz and that

$$\boldsymbol{L} = \boldsymbol{\Gamma} + \boldsymbol{K}\boldsymbol{\Pi} \tag{54}$$

is obtained from the solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ of the regulator equations (42), (43). The combination of eqn. (53) with the observer (52) leads to the feedback controller

$$C: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\hat{\boldsymbol{x}}}_{\rm v}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm v} - \boldsymbol{G}_1\boldsymbol{Q} & -\boldsymbol{G}_1\boldsymbol{C} \\ \boldsymbol{P} - \boldsymbol{G}_2\boldsymbol{Q} + \boldsymbol{B}\boldsymbol{L} & \boldsymbol{A} - \boldsymbol{G}_2\boldsymbol{C} - \boldsymbol{B}\boldsymbol{K} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \\ & & + \begin{pmatrix} \boldsymbol{G}_1 \\ \boldsymbol{G}_2 \end{pmatrix} \boldsymbol{e}(t), \quad \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(0) \\ \hat{\boldsymbol{x}}(0) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v0} \\ \hat{\boldsymbol{x}}_0 \end{pmatrix} \\ \boldsymbol{u}(t) & = & (\, \boldsymbol{L} \quad - \boldsymbol{K} \,) \begin{pmatrix} \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix}, \end{array} \right. \tag{55}$$

which can be abbreviated as

$$C : \begin{cases} \dot{\boldsymbol{x}}_{\rm r}(t) &= \boldsymbol{A}_{\rm r}\boldsymbol{x}_{\rm r}(t) + \boldsymbol{B}_{\rm e}\boldsymbol{e}(t), \quad \boldsymbol{x}_{\rm r}(0) = \boldsymbol{x}_{\rm r0} \\ \boldsymbol{u}(t) &= \boldsymbol{K}_{\rm x}\boldsymbol{x}_{\rm r}(t) \end{cases} \tag{56}$$

to illustrate that it is a dynamical feedback of the control error $\boldsymbol{e}(t)$ towards the control input $\boldsymbol{u}(t)$.

Theorem 2 (Feedback solution of the regulator problem)
Consider the regulator problem under the assumptions (A1) – (A3). There exists a solution $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ to the regulator equations (42), (43) if and only if the condition (51) is satisfied. Then the controller C defined in eqn. (55) solves the regulator problem

- *if the matrix $\boldsymbol{K}$ is chosen so as to make the matrix $\boldsymbol{A} - \boldsymbol{BK}$ Hurwitz,*
- *if the observer feedback $(\boldsymbol{G}_1^{\rm T},\ \boldsymbol{G}_2^{\rm T})^{\rm T}$ makes the state observer (52) asymptotically stable, and*
- *if the feedback matrix $\boldsymbol{L}$ is determined by eqn. (54).*

Then the controller (55) is called a *regulator*.

5.2 Analysis of the closed-loop system

The following investigations should prove Theorem 2 by demonstrating that the closed-loop system satisfies the requirements (i), (ii) and (iv) of the regulator problem. The system consisting of the exosystem (8), the plant (9) and the regulator (55) has the following model:

$$\overline{\Sigma} \circ \Sigma_{\rm v} : \begin{cases} \begin{pmatrix} \dot{\boldsymbol{x}}_{\rm v}(t) \\ \dot{\boldsymbol{x}}(t) \\ \dot{\hat{\boldsymbol{x}}}_{\rm v}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{pmatrix} = \left(\begin{array}{cc|cc} \boldsymbol{A}_{\rm v} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{P} & \boldsymbol{A} & \boldsymbol{BL} & -\boldsymbol{BK} \\ \hline \boldsymbol{G}_1\boldsymbol{Q} & \boldsymbol{G}_1\boldsymbol{C} & \boldsymbol{A}_{\rm v} - \boldsymbol{G}_1\boldsymbol{Q} & -\boldsymbol{G}_1\boldsymbol{C} \\ \boldsymbol{G}_2\boldsymbol{Q} & \boldsymbol{G}_2\boldsymbol{C} & \boldsymbol{P} - \boldsymbol{G}_2\boldsymbol{Q} + \boldsymbol{BL} & \boldsymbol{A} - \boldsymbol{G}_2\boldsymbol{C} - \boldsymbol{BK} \end{array}\right) \begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \\ \\ \boldsymbol{e}(t) = (\, \boldsymbol{Q} \;\; \boldsymbol{C} \mid \boldsymbol{O} \;\; \boldsymbol{O} \,) \begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix}. \end{cases}$$

With the state transformation

$$\begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \\ \Delta\boldsymbol{x}_{\rm v}(t) \\ \Delta\boldsymbol{x}(t) \end{pmatrix} = \underbrace{\left(\begin{array}{cc|cc} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{I} & \boldsymbol{O} & -\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & -\boldsymbol{I} \end{array}\right)}_{\boldsymbol{T}^{-1}} \begin{pmatrix} \boldsymbol{x}_{\rm v}(t) \\ \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}_{\rm v}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \quad \text{with} \quad \boldsymbol{T} = \boldsymbol{T}^{-1},$$

which introduces the state differences

$$\begin{aligned}\Delta\boldsymbol{x}_{\rm v}(t) &= \boldsymbol{x}_{\rm v}(t) - \hat{\boldsymbol{x}}_{\rm v}(t)\\ \Delta\boldsymbol{x}(t) &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t),\end{aligned}$$

the model of the closed-loop system $\overline{\Sigma}$ gets the new form

$$\overline{\Sigma}\circ\Sigma_{\rm v}: \left\{\begin{array}{l} \begin{pmatrix}\dot{\boldsymbol{x}}_{\rm v}(t)\\ \dot{\boldsymbol{x}}(t)\\ \Delta\dot{\boldsymbol{x}}_{\rm v}(t)\\ \Delta\dot{\boldsymbol{x}}(t)\end{pmatrix} = \left(\begin{array}{cc:cc} \boldsymbol{A}_{\rm v} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{P}+\boldsymbol{B}\boldsymbol{L} & \boldsymbol{A}-\boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{L} & \boldsymbol{B}\boldsymbol{K}\\ \hdashline \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{\rm v}-\boldsymbol{G}_1\boldsymbol{Q} & -\boldsymbol{G}_1\boldsymbol{C}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{P}-\boldsymbol{G}_2\boldsymbol{Q} & \boldsymbol{A}-\boldsymbol{G}_1\boldsymbol{C}\end{array}\right)\begin{pmatrix}\boldsymbol{x}_{\rm v}(t)\\ \boldsymbol{x}(t)\\ \Delta\boldsymbol{x}_{\rm v}(t)\\ \Delta\boldsymbol{x}(t)\end{pmatrix}\\ \boldsymbol{e}(t) = (\,\boldsymbol{Q}\;\;\boldsymbol{C}\mid\boldsymbol{O}\;\;\boldsymbol{O}\,)\begin{pmatrix}\boldsymbol{x}_{\rm v}(t)\\ \boldsymbol{x}(t)\\ \hat{\boldsymbol{x}}_{\rm v}(t)\\ \hat{\boldsymbol{x}}(t)\end{pmatrix}\end{array}\right. \tag{57}$$

with the initial state

$$\begin{pmatrix}\boldsymbol{x}_{\rm v}(0)\\ \boldsymbol{x}(0)\\ \Delta\boldsymbol{x}_{\rm v}(0)\\ \Delta\boldsymbol{x}(0)\end{pmatrix} = \begin{pmatrix}\boldsymbol{x}_{\rm v0}\\ \boldsymbol{x}_0\\ \boldsymbol{x}_{\rm v0}-\hat{\boldsymbol{x}}_{\rm v0}\\ \boldsymbol{x}_0-\hat{\boldsymbol{x}}_0\end{pmatrix}.$$

This system satisfied the requirements of the regulator problem as follows:

(i) For $\boldsymbol{x}_{\rm v0}=\boldsymbol{0}$, the representation of $\overline{\Sigma}\circ\Sigma_{\rm v}$ reduces to

$$\overline{\Sigma}\circ\Sigma_{\rm v} \text{ for } \boldsymbol{x}_{\rm v0}=\boldsymbol{0}: \left\{\begin{array}{l} \begin{pmatrix}\dot{\boldsymbol{x}}(t)\\ \Delta\dot{\boldsymbol{x}}_{\rm v}(t)\\ \Delta\dot{\boldsymbol{x}}(t)\end{pmatrix} = \left(\begin{array}{c:cc} \boldsymbol{A}-\boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{L} & \boldsymbol{B}\boldsymbol{K}\\ \hdashline \boldsymbol{O} & \boldsymbol{A}_{\rm v}-\boldsymbol{G}_1\boldsymbol{Q} & -\boldsymbol{G}_1\boldsymbol{C}\\ \boldsymbol{O} & \boldsymbol{P}-\boldsymbol{G}_2\boldsymbol{Q} & \boldsymbol{A}-\boldsymbol{G}_1\boldsymbol{C}\end{array}\right)\cdot\\ \qquad\cdot\begin{pmatrix}\boldsymbol{x}(t)\\ \Delta\boldsymbol{x}_{\rm v}(t)\\ \Delta\boldsymbol{x}(t)\end{pmatrix}\\ \boldsymbol{e}(t) = (\,\boldsymbol{C}\mid\boldsymbol{O}\;\;\boldsymbol{O}\,)\begin{pmatrix}\boldsymbol{x}(t)\\ \hat{\boldsymbol{x}}_{\rm v}(t)\\ \hat{\boldsymbol{x}}(t)\end{pmatrix}.\end{array}\right.$$

This system is asymptotically stable because $\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}$ is Hurwitz and the matrix in the lower right part coincides with the matrix of the observer (52), which is Hurwitz due to the selection of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$. Hence, the requirement (i) of the regulator problem is satisfied.

(ii) To prove that asymptotic regulation occurs, remember that the state observer (52) has the property

$$\lim_{t\to\infty}\begin{pmatrix}\Delta\boldsymbol{x}_{\rm v}(t)\\ \Delta\boldsymbol{x}(t)\end{pmatrix} = \boldsymbol{0}$$

for arbitrary initial states. Hence, the "second part" of the system (57) vanishes and can be deleted for large time t. The remaining part of the model coincides with the system (41), which has been proved to be regulated if the regulator equations (42), (43) are satisfied and $\boldsymbol{L}$ is chosen according to eqn. (54). Hence, the requirement (ii) of the regulator problem is satisfied.

(iv) Due to the condition (51), the regulator equations are well-posed and, thus, solutions exist even if the plant is perturbed. Furthermore, the stabilisation of the plant by the feedback matrix $\boldsymbol{K}$ is robust against parameter perturbations. Hence, the feedback controller (55) provides a robust solution to the regulator problem.

Interpretation. The most interesting point is the fact that the regulator equations (42), (43) are the same for the feedforward solution (40) and for the feedback solution (55). That is, the implementation of the controller in a feedback version to satisfy the robustness requirement (iv) does not introduce any additional constraints. The feedback controller includes the model of the exosystem, which is referred to as the *internal-model principle*.

The feedback controller given above is not the only solution to the regulator problem. If the existence condition (51) of a regulator is satisfied, several solutions may be used. In particular, if the plant is asymptotically stable, the matrix $\boldsymbol{K} = \boldsymbol{O}$ satisfies the conditions stated in Theorem 2 and the observer for the plant state can be removed. Furthermore, it is well-known that the dynamical order of state observers can be reduced or replaced by reduced-order observers. If the plant is not asymptotically stable, an output feedback $\boldsymbol{u}(t) = -\boldsymbol{K}_{\mathrm{y}}\boldsymbol{y}(t)$ may exist to stabilise it.

5.3 Regulator design algorithm

The controller design is summarised in Algorithm 1.

Algorithm 1 *Regulator design*

Given:	Plant Σ_{S} satisfying assumptions (A2), (A3) Exosystem Σ_{v} satisfying assumption (A1).
1.	Find a stabilising state-feedback $\boldsymbol{K}$ such that $\boldsymbol{A} - \boldsymbol{BK}$ is Hurwitz.
2.	Solve the regulator equations (42), (43) to get the controller matrix $\boldsymbol{L} = \boldsymbol{\Gamma} + \boldsymbol{K\Pi}$.
3.	Find feedback matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ to solve the state-observation problem.
4.	Set-up the feedback controller (55).
Result:	Feedback controller (55) that solves the regulator problem.

Steps 1 and 3 can be accomplished with the help of well-known feedback and observer design methods [13]. In Step 2 the way of solution to the regulator equations given in Section 4.4 can be used.

6 Example

Consider the problem to move a point mass on a circle. The two state variables of the exosystem $\Sigma_{\rm v}$ with

$$\boldsymbol{A}_{\rm v} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \boldsymbol{x}_{\rm v0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

describes the required x-coordinate and y-coordinate of the mass at time t. The vector $\boldsymbol{x}_{\rm v}(t)$ rotates around the unit circle in the x/y-plane with the frequency ω.

The point mass gets the two forces $u_1(t) = f_{\rm x}(t)$ and $u_2(t) = f_{\rm y}(t)$ that act in the two coordinate directions and the plant $\Sigma_{\rm S}$ has the model (9) with

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix}, \quad \boldsymbol{P} = \boldsymbol{O}, \quad \boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \\ y(t) \\ \dot{y}(t) \end{pmatrix} \\ \boldsymbol{C} &= -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \boldsymbol{I} \end{aligned}$$

with m denoting the mass. The following parameters are used:

$$m = 10\,\mathrm{kg}, \quad \omega = 1\,\frac{\mathrm{rad}}{\mathrm{s}}$$

when measuring the distances in meters and the time in seconds. The point mass starts in the position

$$\boldsymbol{x}_0 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ -\frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}.$$

Theorem 1 leads to the following design steps. First, a state-feedback controller

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t)$$

should be found such that the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ is Hurwitz. For the eigenvalues

$$\lambda_1 = -0.25, \quad \lambda_2 = -0.4, \quad \lambda_3 = -0.5, \quad \lambda_4 = -0.6$$

in the set `Lambda` the matrix

```
» K=place(A, B, Lambda);
K=    1.9561   9.0349   -0.5597   -1.2167
     -0.5600   -1.2175   1.6939    8.4651
```

is found. The solution to the regulator equations is obtained as described in Section 4.4:

```
Pi =
  1    0
  0   -1
  0    1
  1    0

Gamma =
  -10    0
   0   -10
```

Then the controller (44) is used with these parameters. The top part of Fig. 8 shows that the two components of the control error $e(t)$ vanish within around 20 seconds. The lower part of the figure depicts the x-coordinate with the dashed line indicating the command signal generated by the exosystem. Obviously, the regulator problem is solved (without the robustness requirement (iv)).

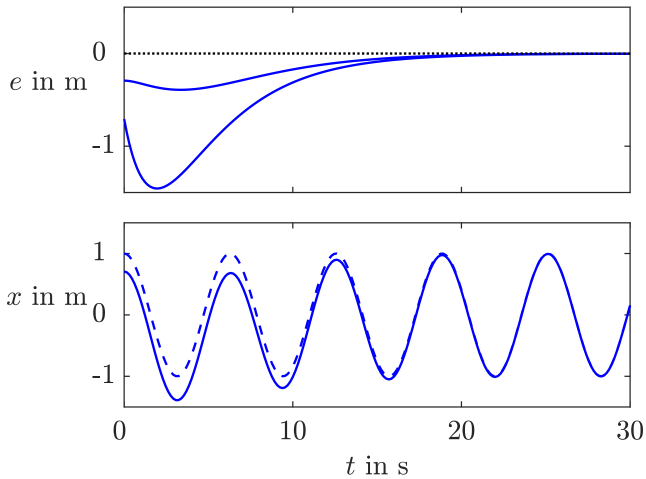

Fig. 8: Behaviour of the point mass with the feedforward controller (44)

To show that this controller is sensitive to parameter variations, the mass is increased to 13 kg and the result shown in Fig. 9. After the parameter changes, command tracking can no longer be reached.

Now, the application of the feedback solution according to Theorem 2 is demonstrated. According to Algorithm 1, after the design steps 1 and 2, which have already been accomplished, an observer should be found to reconstruct the state vectors of the exosystem and of the point mass. For the observer (52) the feedback matrix

```
G =
  -1.8409   -3.9952
   5.3103   -1.1427
   2.4799   -4.7804
  -0.2947   -2.2310
   6.4637    2.8364
   2.9083    0.4202
```

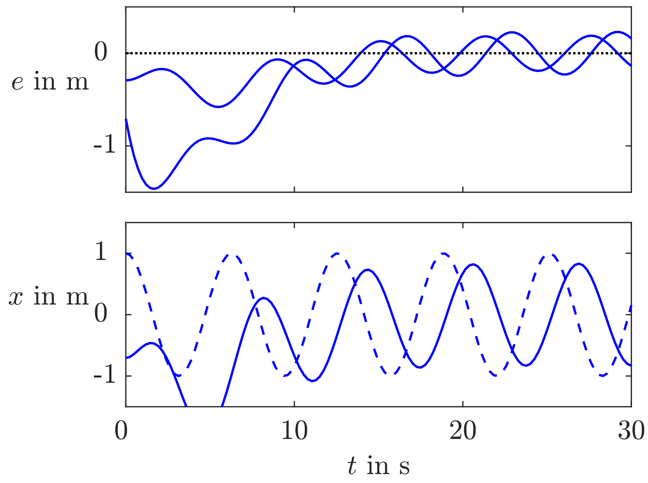

Fig. 9: Behaviour with increased mass

moves the eigenvalues of the observer to the set

```
» LambdaBeob=[-1 -1.2 -1.3 -1.5 -1.6 -1.7];
```

The controller (55) leads to the behaviour of the point mass shown in Fig. 10, where the state observer had a zero initial state. The behaviour is similar to that of the point mass with the feedforward controller but now the controller is robust against parameter variations. If, again, the mass is increased, a similar behaviour results, as Fig. 11 shows.

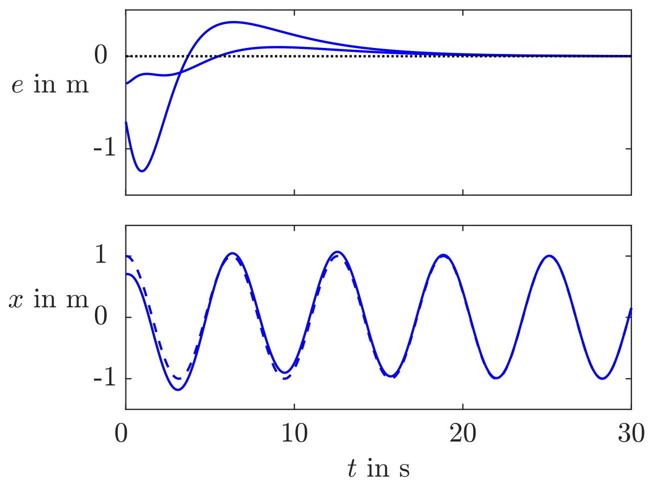

Fig. 10: Behaviour of the point mass with feedback controller

7 References and extensions

The earliest and most cited references concerning the regulator problems are [2, 3, 4] and [5]. There are only a few monographs that explain this problem in detail like [9] (Chapter 1),

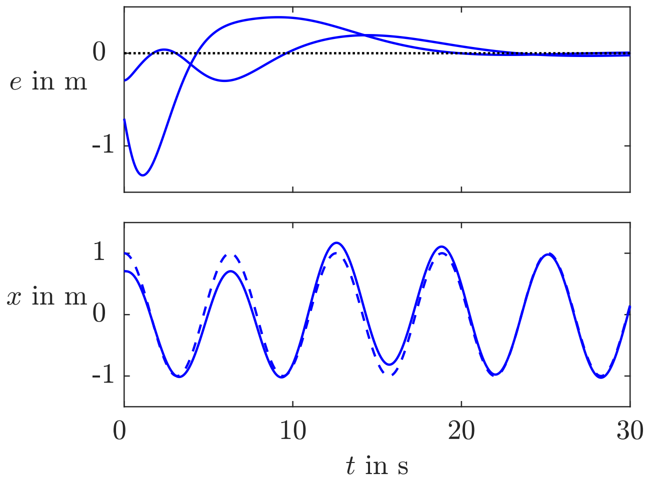

Fig. 11: Point mass with feedback controller with $m = 13\,\mathrm{kg}$

[11] (Chapter 4) and [16] (Chapter 9).

The regulator equations have several consequences, which have not been mentioned in this report. First, they lead to the Internal-Model Principle requiring that any feedback solution to the regulator problem has to include the model of the exosystem [7, 8]. In the feedback solution given in Section 5 this model is part of the Luenberger observer. Second, an analysis of the controlled plant shows that zeros play an important role for the signal transmission from the exosystem through the controlled plant towards the control output to satisfy the requirement (ii) (cf. [6] and the discussion in Example 2).

As mentioned in Section 4.4, the solution to the regulator equations is unique if the plant has the same number of inputs and outputs. Otherwise linear-equation solvers find a solution that is optimal in some error-norm sense. A detailed analysis of the latter situation has been published in [1]. Accordingly, if the control input is underdetermined, one may use an energy-efficient input, whereas if the regulator equations do not have any solution the "most accurate" input with respect to some performance criterion should be used.

An extension of the output regulation problem to multi-agent systems with agents having identical dynamics has been obtained in [10, 18] and generalised to agents with individual dynamics in [15]. The robustness properties of these solutions have been the topic of [17].

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Appendix A: Preliminary results of matrix vectorisation

The operation $\mathrm{vec}(.)$ is used to reshape a matrix as a vector in which all the columns of the matrix are written one below the other. In the definition of this operation

$$\mathrm{vec}(\boldsymbol{\Pi}) = \begin{pmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_{n_\mathrm{v}} \end{pmatrix} \quad \text{for} \quad \boldsymbol{\Pi} = (\boldsymbol{\pi}_1\, \boldsymbol{\pi}_2 \ldots \boldsymbol{\pi}_{n_\mathrm{v}})$$

the vectors $\boldsymbol{\pi}_i$, $(i = 1, 2, ..., n_\mathrm{v})$ are the columns of the matrix $\boldsymbol{\Pi}$.

The vectorisation of the product $\boldsymbol{A}\boldsymbol{\Pi}$ can be represented in terms of the vectorisation of the matrix $\boldsymbol{\Pi}$ as

$$\mathrm{vec}(\boldsymbol{A}\boldsymbol{\Pi}) = (\boldsymbol{I} \otimes \boldsymbol{A}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) \tag{A.1}$$

with $\otimes$ denoting the Kronecker product. To understand this relation, consider the following transformations:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{\Pi} &= \boldsymbol{A}\,(\boldsymbol{\pi}_1\, \boldsymbol{\pi}_2 \ldots \boldsymbol{\pi}_{n_\mathrm{v}}) \\ &= (\boldsymbol{A}\boldsymbol{\pi}_1\; \boldsymbol{A}\boldsymbol{\pi}_2\; \ldots\; \boldsymbol{A}\boldsymbol{\pi}_{n_\mathrm{v}}) \\ \mathrm{vec}(\boldsymbol{A}\boldsymbol{\Pi}) &= \mathrm{vec}(\boldsymbol{A}\boldsymbol{\pi}_1\; \boldsymbol{A}\boldsymbol{\pi}_2\; \ldots\; \boldsymbol{A}\boldsymbol{\pi}_{n_\mathrm{v}}) \\ &= \begin{pmatrix} \boldsymbol{A} & & & \\ & \boldsymbol{A} & & \\ & & \ddots & \\ & & & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_{n_\mathrm{v}} \end{pmatrix} \\ &= (\boldsymbol{I} \otimes \boldsymbol{A}) \cdot \mathrm{vec}(\boldsymbol{\Pi}) \end{aligned}$$

Likewise, the vectorisation of the product $\boldsymbol{\Pi}\boldsymbol{A}_\mathrm{v}$ can be written in terms of $\mathrm{vec}(\boldsymbol{\Pi})$ as

$$\mathrm{vec}(\boldsymbol{\Pi}\boldsymbol{A}_\mathrm{v}) = \left(\boldsymbol{A}_\mathrm{v}^\mathrm{T} \otimes \boldsymbol{I}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi}), \tag{A.2}$$

because the relations

$$\begin{aligned} \boldsymbol{\Pi}\boldsymbol{A}_\mathrm{v} &= (\boldsymbol{\pi}_1\, \boldsymbol{\pi}_2 \ldots \boldsymbol{\pi}_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\ &= \left(\sum_{j=1}^n a_{j1}\boldsymbol{\pi}_j \;\; \sum_{j=1}^n a_{j2}\boldsymbol{\pi}_j \;\ldots\; \sum_{j=1}^n a_{jn}\boldsymbol{\pi}_j \right) \\ \mathrm{vec}(\boldsymbol{\Pi}\boldsymbol{A}_\mathrm{v}) &= \begin{pmatrix} \sum_{j=1}^n a_{j1}\boldsymbol{\pi}_j \\ \sum_{j=1}^n a_{j2}\boldsymbol{\pi}_j \\ \vdots \\ \sum_{j=1}^n a_{jn}\boldsymbol{\pi}_j \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a_{11}\boldsymbol{I} & a_{21}\boldsymbol{I} & \cdots & a_{n1}\boldsymbol{I} \\ a_{12}\boldsymbol{I} & a_{22}\boldsymbol{I} & \cdots & a_{n2}\boldsymbol{I} \\ \vdots & \vdots & & \vdots \\ a_{1n}\boldsymbol{I} & a_{2n}\boldsymbol{I} & \cdots & a_{nn}\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_n \end{pmatrix}$$

$$= \left(\boldsymbol{A}_{\rm v}^{\rm T} \otimes \boldsymbol{I}\right) \cdot \mathrm{vec}(\boldsymbol{\Pi})$$

hold.

Appendix B: Well-posedness of linear equations

This appendix summarises results on the well-posedness of linear equations as stated in [16], Section 9.2. A mathematical problem is called *well-posed* if it has a solution and it remains solvable after a small perturbation of the data occurring in the problem.

The linear equation

$$\boldsymbol{F}\boldsymbol{x} = \boldsymbol{b} \qquad \text{(B.1)}$$

should be solved for a given $(m \times n)$–matrix $\boldsymbol{F}$ and an m–vector $\boldsymbol{b}$ for the n–vector $\boldsymbol{x}$. It is well known that a solution exists if the vector $\boldsymbol{b}$ belongs to the image of $\boldsymbol{F}$

$$\boldsymbol{b} \in \mathrm{im}(\boldsymbol{F}) \qquad \text{(B.2)}$$

or, equivalently, if the following condition is satisfied:

$$\mathrm{rank}\,(\boldsymbol{F}) = \mathrm{rank}\,(\boldsymbol{F}\ \boldsymbol{b}).$$

If $\boldsymbol{F}$ is decomposed into its column vectors

$$\boldsymbol{F} = (\boldsymbol{f}_1\ \boldsymbol{f}_2 \dots \boldsymbol{f}_n)$$

the last two equations say that $\boldsymbol{b}$ can be represented as a linear combination

$$\boldsymbol{b} = \sum_{i=1}^{n} x_i \boldsymbol{f}_i$$

of these vectors, in which the elements x_i of the vector $\boldsymbol{x}$ appear as coefficients. Consequently, eqn. (B.2) is a necessary and sufficient condition for the linear equation (B.1) to have a solution.

In the general case, a small perturbation of the vector $\boldsymbol{b}$ can destroy the solvability of eqn. (B.1) and, hence, this equation is not well-posed. If the rank of $\boldsymbol{F}$ is less than m

$$\mathrm{rank}\,(\boldsymbol{F}) < m,$$

then the image of $\boldsymbol{F}$ is a subset of $\mathrm{I\!R}^m$ and only specific m–vectors $\boldsymbol{b}$ satisfy the requirement (B.2). However, a small perturbation generally moves these specific vectors $\boldsymbol{b}$ out of $\mathrm{im}(\boldsymbol{F})$.

In contrast, if

$$\mathrm{rank}\,(\boldsymbol{F}) = m$$

holds and, consequently,

$$\mathrm{im}(\boldsymbol{F}) = \mathrm{I\!R}^m,$$

any m–vector $\boldsymbol{b}$ belongs to $\mathrm{im}(\boldsymbol{F})$ and, hence, the linear equation (B.1) has a solution for all perturbations of $\boldsymbol{b}$. This result is summarised in the following lemma:

Lemma 5 *The linear equation $\boldsymbol{F}\boldsymbol{x} = \boldsymbol{b}$, which is to be solved for the vector $\boldsymbol{x}$, is well-posed if and only if*

$$\mathrm{rank}\,(\boldsymbol{F}) = m. \tag{B.3}$$

As a consequence, one has to distinguish between the solvability of the linear equation (B.1) for specific $\boldsymbol{F}$ and $\boldsymbol{b}$ (also called the *individual solvability*) and the solvability for general $\boldsymbol{F}$ and $\boldsymbol{b}$ (called the *universal solvability*). The well-posedness of the linear equation concerns the universal solvability and requires the matrix $\boldsymbol{F}$ to possess the property (B.3).

As the regulator problem includes the robustness requirement (iv), the well-posedness of the Sylvester equation and of the regulator equations are required, which make the conditions given in the main part of this report to be necessary, although for specific systems violating these conditions a solution of the regulator equations may exist.