Errata

The notion of a primitive matrix has been misinterpreted in Theorem 3.3 and in the appendix on matrices. The correct passages of the text are attached.

Theorem 3.3 (Necessary and sufficient condition for discrete-time consensus)
Consider a multi-agent system $\overline{\Sigma}$ described by eqn. (3.52) with a row-stochastic matrix $\boldsymbol{P}$. The overall system $\overline{\Sigma}$ reaches a consensus as in eqn. (3.54) for some consensus value $\bar{x} \in \mathbb{R}$ if and only if the matrix $\boldsymbol{P}$ has only the eigenvalue $\lambda_1 = 1$ with modulus 1:

$$|\lambda_i| < 1, \quad i = 2, 3, ..., N.$$

In particular, if $\boldsymbol{P}$ is irreducible and has positive diagonal elements

$$p_{ii} > 0, \quad i = 1, 2, ..., N \tag{3.56}$$

it is primitive and satisfies the condition of the theorem (cf. Lemma A2.4). Then the induced graph $\vec{\mathcal{G}}(\boldsymbol{P})$ has self-loops at any vertex, which is a reasonable communication structure of the agents, because $p_{ii} > 0$ means that the state x_i of the i-th agent at time $k+1$ depends upon this state at time k. Hence, the iteration (3.53) improves the state x_i while using its earlier values. For $p_{ii} = 0$, $x_i(k+1)$ were a linear combination only of the neighbouring states and the old value $x_i(k)$ were forgotten.

Proof of Theorem 3.3. The theorem can be proved similarly to Theorem 3.1 with the notation introduced there. If the state transformation (3.11) in its discrete-time version

$$\underbrace{\begin{pmatrix} x_1(k) \\ e_2(k) \\ \vdots \\ e_N(k) \end{pmatrix}}_{\tilde{\boldsymbol{x}}} = \underbrace{\left(\begin{array}{c|c} 1 & \boldsymbol{0}^{\mathrm{T}} \\ \hline -\boldsymbol{1} & \boldsymbol{I} \end{array}\right)}_{\boldsymbol{T}^{-1}} \begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{pmatrix}$$

is applied to the system (3.52) the new representation

$$\tilde{\boldsymbol{x}}(k+1) = \boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T}\,\tilde{\boldsymbol{x}}(k) \tag{3.57}$$

includes the transformed matrix $\boldsymbol{P}$, which, by means of eqn. (3.55), is decomposed as

$$\boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T} = \left(\begin{array}{c|c} 1 & \boldsymbol{p}_{12}^{\mathrm{T}} \\ \hline \boldsymbol{0} & \tilde{\boldsymbol{P}}_{22} \end{array}\right) \quad \text{with} \quad \tilde{\boldsymbol{P}}_{22} = \boldsymbol{P}_{22} - \boldsymbol{1}\,\boldsymbol{p}_{12}^{\mathrm{T}}. \tag{3.58}$$

The eigenvalues of $\tilde{\boldsymbol{P}}_{22}$ coincide with the eigenvalues λ_2,..., λ_N of $\boldsymbol{P}$, but they do not include $\lambda_1 = 1$. For the state differences $e_i(k) = x_i(k) - x_1(k)$, $(i = 2, 3, ..., N)$ the model (3.57) yields

$$\begin{pmatrix} e_2(k+1) \\ e_3(k+1) \\ \vdots \\ e_N(k+1) \end{pmatrix} = \tilde{\boldsymbol{P}}_{22} \begin{pmatrix} e_2(k) \\ e_3(k) \\ \vdots \\ e_N(k) \end{pmatrix}, \qquad \begin{pmatrix} e_2(0) \\ e_3(0) \\ \vdots \\ e_N(0) \end{pmatrix} = \begin{pmatrix} x_{20} - x_{10} \\ x_{30} - x_{10} \\ \vdots \\ x_{N0} - x_{10} \end{pmatrix}. \tag{3.59}$$

All differences $e_i(k)$ vanish asymptotically for all initial states $e_i(0)$ if and only if all eigenvalues of $\tilde{\boldsymbol{P}}_{22}$ have modulus less than one as required by the theorem. Hence, all agent states x_i become equal:

$$e_i(k) \stackrel{k\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad |x_i(k) - x_1(k)| \stackrel{k\to\infty}{\longrightarrow} 0, \quad i = 2, 3, ..., N.$$

and a consensus is reached as stated in eqn. (3.54). □

  – *the spectral radius is positive:* $\lambda_{\mathrm{P}} > 0$,
  – *there exists a positive vector* $\boldsymbol{v} > \boldsymbol{0}$ *such that* $\boldsymbol{A}\boldsymbol{v} = \lambda_{\mathrm{P}}\boldsymbol{v}$ *holds,*
  – $\lambda = \lambda_{\mathrm{P}}$ *is a simple eigenvalue and for all other eigenvalues the inequality* $|\lambda| \le \lambda_{\mathrm{P}}$ *holds.*

- *If* $\boldsymbol{A}$ *is nonnegative*
  – *the spectral radius is nonnegative:* $\lambda_{\mathrm{P}} \ge 0$,
  – *there exists a nonnegative vector* $\boldsymbol{v} \ge \boldsymbol{0}$, $\boldsymbol{v} \ne \boldsymbol{0}$ *such that* $\boldsymbol{A}\boldsymbol{v} = \lambda_{\mathrm{P}}\boldsymbol{v}$ *holds.*

Hence, for positive and for irreducible nonnegative matrices the spectral radius represents a unique real eigenvalue of $\boldsymbol{A}$, which has the maximum modulus. The main difference between positive and general nonnegative matrices lies in the property of the corresponding eigenvector $\boldsymbol{v}$, which is positive or nonnegative, respectively.

A nonnegative matrix is called *primitive* if there exists an integer m such that $\boldsymbol{A}^m > 0$ holds. Primitive matrices have a single eigenvalue, which is equal to the Perron root, and all other eigenvalues have a smaller magnitude.

A necessary condition for $\boldsymbol{A} \ge 0$ to be primitive is that $\boldsymbol{A}$ has to be irreducible, which is equivalent to the requirement that the directed graph $\vec{\mathcal{G}}(\boldsymbol{A})$ is strongly connected. However, this condition is only necessary, but not sufficient for $\boldsymbol{A}$ to be primitive. If, in addition to the strong connectedness of the graph all diagonal elements a_{ii}, $(i = 1, 2, ..., n)$ of $\boldsymbol{A}$ are positive, then $\boldsymbol{A}$ is primitive.

In summary, for nonnegative $(n \times n)$–matrices the following implications hold:

$\boldsymbol{A}$ is positive	$\Longrightarrow$	$\boldsymbol{A}$ is primitive	$\Longrightarrow$	$\boldsymbol{A}$ is irreducible
$\boldsymbol{A} > 0$		$\exists k : \ \boldsymbol{A}^k > 0$		$\sum_{k=0}^{n-1} \boldsymbol{A}^k > 0.$

A2.4.8 Stochastic matrices

An $(n \times n)$–matrix $\boldsymbol{P}$ is called stochastic, if it is nonnegative and all row sums equal one:

$$\boldsymbol{P} \ge 0, \quad \boldsymbol{P}\boldsymbol{1} = \boldsymbol{1}. \tag{A2.12}$$

Such matrices are also said to be a row-stochastic or to be a probability matrix or a Markov matrix. A doubly-stochastic matrix is a nonnegative matrix with both row sum and column sum equal to one.

Stochastic matrices have an eigenvalue $\lambda_1 = 1$ and eqn. (A2.12) yields the eigenvector $\boldsymbol{v} = \boldsymbol{1}$ belonging to λ_1. If the matrix is doubly-stochastic, $\boldsymbol{w}^{\mathrm{T}} = \boldsymbol{1}^{\mathrm{T}}$ is a left eigenvector for the eigenvalue λ_1. More generally, Gershgorin's theorem (Lemma A2.2) implies that all eigenvalues lie in the unit circle:

$$|\lambda_i\{\boldsymbol{P}\}| \le 1, \quad i = 1, 2, ..., n.$$

In order to ensure the convergence of the consensus problem (3.52), the eigenvalue λ_1 of the matrix $\boldsymbol{P}$ has to be the only eigenvalue with magnitude $|\lambda_i| = 1$. A stochastic matrix with this property is called ergodic.

Lemma A2.4 (Primitive stochastic matrices)
Any irreducible stochastic matrix $\boldsymbol{P}$ *with positive diagonal elements is primitive.*

Products of stochastic matrices are stochastic. The limit

$$\lim_{k\to\infty} \boldsymbol{P}^k = \boldsymbol{P}_\infty \tag{A2.13}$$

exists if $\boldsymbol{P}$ is primitive. The matrix $\boldsymbol{P}_\infty$ consists of n identical rows $\bar{\boldsymbol{p}}^{\mathrm{T}}$:

$$\boldsymbol{P}_\infty = \begin{pmatrix} \bar{\boldsymbol{p}}^{\mathrm{T}} \\ \bar{\boldsymbol{p}}^{\mathrm{T}} \\ \vdots \\ \bar{\boldsymbol{p}}^{\mathrm{T}} \end{pmatrix}.$$

Hence, a consensus system $\boldsymbol{x}(k+1) = \boldsymbol{P}\boldsymbol{x}(k)$ converges to the consensus state $\lim_{k\to\infty} \boldsymbol{x}(k) = \bar{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{x}_0$.

A2.5 Stability analysis of complex-valued matrices

A complex $(n \times n)$–matrix $\boldsymbol{C}$ is called Hurwitz if all eigenvalues have negative real part:

$$\mathrm{Re}\,(\lambda_i\{\boldsymbol{C}\}) < 0, \quad i = 1, 2, ..., n.$$

For the stability analysis of complex matrices consider the characteristic polynomial

$$p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{C}) = \sum_{i=0}^{n} a_i \lambda^i$$

with the complex coefficients $a_i = p_i + \mathrm{j}q_i$, $(i = 0, 1, ..., n)$. a_i^* denotes the conjugate complex of a_i. The following theorem is a generalisation of the well-known Hurwitz criterion:

Lemma A2.5 (Bilharz criterion) [42]
The roots λ_i, $(i = 1, 2, ..., n)$ of the polynomial $p(\lambda)$ satisfy the condition $\mathrm{Re}(\lambda_i) < 0$ if and only if all the principal minors of even order of the $(2n \times 2n)$–matrix

$$\boldsymbol{H} = \begin{pmatrix} a_0 & -\mathrm{j}a_0^* & 0 & 0 & 0 & \cdots \\ -\mathrm{j}a_1 & a_1^* & a_0 & -\mathrm{j}a_0^* & 0 & \cdots \\ -a_2 & \mathrm{j}a_2^* & -\mathrm{j}a_1 & a_1^* & a_0 & \cdots \\ \mathrm{j}a_3 & -a_3^* & -a_2 & \mathrm{j}a_2^* & -\mathrm{j}a_1 & \cdots \\ a_4 & -\mathrm{j}a_4^* & \mathrm{j}a_3 & -\mathrm{j}a_3^* & -a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

are positive:

$$D_i > 0, \quad i = 2, 4, ..., 2n.$$

In the matrix $\boldsymbol{H}$, elements a_i with $i > n$ are set to zero.