7

Generic properties of linear systems

Controllability and observability are generic properties of dynamical systems that can be found by using the structure graph. Whereas most of the control problems require that the plant has these generic properties, three examples illustrate that in specific situations the violation of generic properties due to a specific parameter adjustment helps to satisfy the control goals.

7.1 Generic properties and atypical systems

This chapter deals with properties of linear dynamical systems that are established by the system structure and hold for a large set of parameter values. These properties are not only valid for specific systems, but they hold for large sets of systems and are, thus, said to be *generic properties* (or structural properties). The controllability and the observability, which will be investigated in this chapter in detail, are two important examples of generic properties, both of which are prerequisites for solving control problems. It will be explained how they can be expressed in graph-theoretical terms and in which sense they are generic.

The chapter has two goals. First, it should describe properties of dynamical systems that can be found without knowing the parameter values of a system Σ under consideration. Instead of a state-space model with a fixed parameter set, a directed graph is used for the analysis, which will be called the *structure graph* of the system. To set up this graph, the only information necessary concerns the position of the non-zero elements in the matrices of the state-space model of Σ. These positions are the same for a large set $\mathcal{S}$ of systems and, hence, typical for the system class that Σ belongs to. Graph-theoretical methods are used to check whether this system class possesses the generic properties of controllability and observability.

The second goal is to reveal the general methodology how to analyse structural models and how to come to conclusions with respect to specific systems. Since the structure graph does not represent an individual system Σ but a class $\mathcal{S}$ of structurally equivalent systems, the well known properties of controllability and observability cannot be applied. New notions have to be introduced, which will be called *structural controllability* and *structural observability*. To clarify the meaning of the new notions, a relation is set up between the properties of the individual systems and the structural properties.

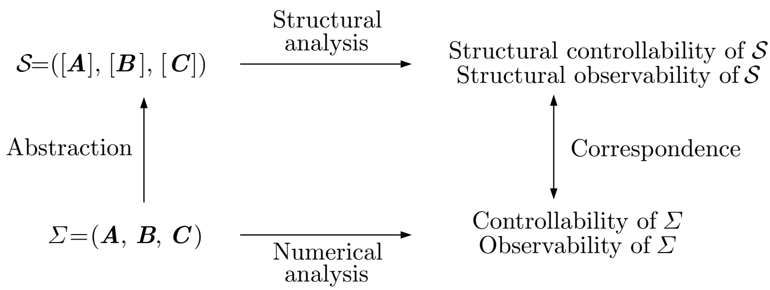

Fig. 7.1: Ways of analysing the controllability and observability of a system

To emphasise the importance of the second goal of this chapter, Fig. 7.1 compares two ways for analysing a system Σ. The usual way, which is called numerical analysis, takes a system $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ and uses the well-known criteria of KALMAN or HAUTUS to check whether Σ is completely controllable and completely observable. Since both criteria provide necessary and sufficient conditions, this check gives a unique result.

The structural analysis distinguishes from the numerical analysis in two points, namely with respect to the subject of the analysis and with respect to the interpretation of the result. It does not consider a single system Σ, but the class $\mathcal{S}$ of systems that have the same structure as Σ and is described by the structure matrices $[\boldsymbol{A}]$, $[\boldsymbol{B}]$ and $[\boldsymbol{C}]$. The analysis result is a characterisation of the structural observability and the structural controllability of the set $\mathcal{S}$. Necessary and sufficient conditions for these properties will be given below, which imply that the structural analysis, like the numerical analysis, gives the best possible result in deciding whether the system class $\mathcal{S}$ has these properties or not.

Figure 7.1 points to three important questions, which will be answered in this chapter:

- How should the structural controllability and the structural observability be defined?
- How can a class $\mathcal{S}$ of systems be analysed with respect to these structural properties?
- What is the correspondence between the numerical and the structural properties of controllability and observability?

The first question is answered by Definitions 7.2 and 7.3, which also imply the answer to the last question given by Corollary 7.1 stating that the structural properties of a class of systems are necessary for the numerical properties to exist for any system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathcal{S}$ of this class.

Necessary and sufficient conditions for the structural properties are given by Theorems 7.1 and 7.2 as answers to the second question.

Validity of generic properties. With respect to the correspondence marked on the right-hand side of Fig. 7.1 the question occurs how do the structural properties of the set $\mathcal{S}$ transfer to any individual system $\Sigma \in \mathcal{S}$. More precisely: If $\mathcal{S}$ is structurally controllable, how many systems $\Sigma \in \mathcal{S}$ are controllable? The answer is: *almost all*.

The term "almost all" has a clear mathematical meaning. It is used if an infinite set is considered and one has a property Π that holds true for all elements in an open and dense subset. Then every point of this infinite set has this property Π or is arbitrarily close to such a point. Without going into details, it is important to understand that such a property Π is generic in the sense that nearly all elements have it and the elements of the set that do not have it are *atypical* representatives of the set.

In mathematical terms, consider a class of systems with the parameters p_1, p_2,..., p_d, which vary independently of each other. Consider a property Π that is valid for all systems *unless* the polynomials $\phi_i(\boldsymbol{p})$, $(i = 1, 2, ..., l)$ of the parameter vector $\boldsymbol{p} \in \mathrm{IR}^d$ vanish. Then the set

$$\mathcal{P} = \{\boldsymbol{p} \in \mathrm{IR}^d \mid \phi_1(\boldsymbol{p}) = \phi_2(\boldsymbol{p}) = ... = \phi_l(\boldsymbol{p}) = 0\},$$

for which the property Π does *not* hold is an *algebraic variety* (Appendix A2.1 on p. 720). The important aspect is the fact that $\mathcal{P}$ is a closed set. If a property Π does not hold for some parameter vector $\boldsymbol{p} \in \mathcal{P}$ then a small change of $\boldsymbol{p}$ most likely moves the parameter vector out of the set $\mathcal{P}$. The interpretation is that the property Π is generic and almost all systems with a parameter $\boldsymbol{p} \in \mathrm{IR}^d$ have it.

Example 7.1 *Generic rank of a matrix and the set of atypical matrices*

To illustrate this fact, consider the matrix and its structure matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \quad \rightarrow \quad [\boldsymbol{A}] = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

$[\boldsymbol{A}]$ has the structural rank equal to 3, which implies that the matrix $\boldsymbol{A}$ has the numerical rank 3 for "almost all" real elements a_{11}, a_{22}, a_{33}, a_{13} and a_{24} (cf. Example 5.8).

A detailed analysis of $\boldsymbol{A}$ shows that this matrix has full rank if and only if the relation

$$a_{11}a_{22}a_{33} \neq 0$$

holds. Stated the other way round, $\boldsymbol{A}$ has not the full rank if and only if the relation

$$a_{11}a_{22}a_{33} = 0 \tag{7.1}$$

is valid. Equation (7.1) describes the exceptional parameter set, for which the rank of $\boldsymbol{A}$ is smaller than the structural rank of $[\boldsymbol{A}]$, in terms of the polynomial

$$\phi_1(a_{11}, a_{22}, a_{33}) = a_{11}a_{22}a_{33}.$$

As the element a_{13} does not matter in this investigation, Fig. 7.2 shows the parameter space IR^3 of the three remaining elements. The exceptional parameters for which the generic property is not a property of $\boldsymbol{A}$ lie on the three planes, which are described by the relations

$$a_{11} = 0, \quad a_{22} = 0 \quad \text{or} \quad a_{33} = 0.$$

The set of all these parameters

$$\mathcal{P} = \left\{ \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \end{pmatrix} \in \mathrm{IR}^3 \;\middle|\; \phi_1(a_{11}, a_{22}, a_{33}) = 0 \right\}$$

is an algebraic variety according to Appendix A2.1.

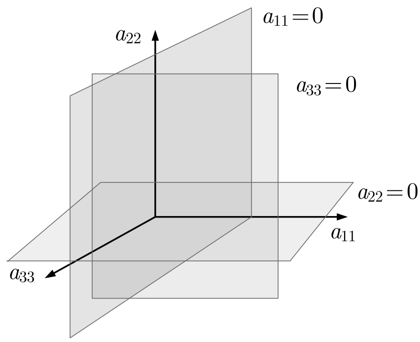

Fig. 7.2: Parameter space of the matrix considered with three planes of exceptional parameter values

This example illustrates that a property, which can be recognised by considering the structure of a matrix (or, more generally, the structure of a system), is generic if it is transferred to any matrix or system of an infinite set with the exception of some elements. The set of exceptions can be visualised as a hypersurface in the parameter space. If a change of one or several parameters move the matrix (or system) away from the hypersurface, the generic property appears. □

7.2 Results on the controllability and observability of linear systems

7.2.1 Models

This section reviews well known results concerning the controllability and observability of linear systems as a preparation of the structural investigations of the next sections. The system

$$\Sigma: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \end{cases} \tag{7.2}$$

has the state vector $\boldsymbol{x} \in \mathrm{IR}^n$, the input vector $\boldsymbol{u} \in \mathrm{IR}^m$ and the output vector $\boldsymbol{y} \in \mathrm{IR}^p$. The output feedback

$$C: \quad \boldsymbol{u}(t) = -\boldsymbol{K}_{\mathrm{y}}\boldsymbol{y}(t) \tag{7.3}$$

has the $(m \times p)$–matrix $\boldsymbol{K}_{\mathrm{y}}$ and leads to the closed-loop system (7.2), (7.3) represented by

$$\overline{\Sigma}: \begin{cases} \dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C})\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{cases} \tag{7.4}$$

with the closed-loop system matrix

$$\overline{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C}. \tag{7.5}$$

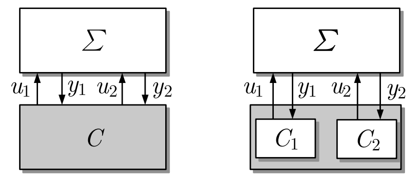

Fig. 7.3: Centralised vs. decentralised control of a system Σ

Structurally constrained controllers. Most of the investigations of this chapter concern the situation in which the controller can combine any output $y_i(t)$, $(i = 1, 2, ..., p)$ with any input $u_j(t)$, $(j = 1, 2, ..., m)$, which is the typical situation in multivariable control. In some applications, however, the controller has to be restricted, for example, to be a decentralised controller that consists of independent feedback loops from the i–th output $y_i(t)$ towards the i–th input $u_i(t)$ and is represented by separate equations

$$C_i: \quad u_i(t) = -k_{ii} y_i(t), \quad i = 1, 2, ..., N \tag{7.6}$$

for the $N = m = p$ control channels (u_i, y_i). Then the controller (7.3) has a diagonal matrix:

$$\boldsymbol{K}_{\rm y} = \begin{pmatrix} k_{11} & & & \\ & k_{22} & & \\ & & \ddots & \\ & & & k_{NN} \end{pmatrix}.$$

Figure 7.3 compares both situations. In the left part, C is said to be a *centralised controller*, because all output signals are processed by a centralised control unit and fed back to all inputs. In the right part of the figure the controller is restricted to combine only the output with the input of the i–th channel. This controller is said to be a *decentralised controller*. Its control stations (7.6) can be implemented in a locally distributed manner.

7.2.2 Controllability and observability criteria

The controllability concerns the situation that a system is at time $t = 0$ in the state $\boldsymbol{x}_0$ and should be moved within a finite time interval $[0, t_{\rm e}]$ into an arbitrarily given final state $\boldsymbol{x}(t_{\rm e}) = \boldsymbol{x}_{\rm e}$ by an appropriately chosen control input $\boldsymbol{u}_{[0,t_{\rm e}]}$. The symbol $\boldsymbol{u}_{[0,t_{\rm e}]}$ emphasises that it is not sufficient to select $\boldsymbol{u}(t)$ at a specific time t, but one has to choose the input over the indicated time period $0 \le t \le t_{\rm e}$. The question to be answered asks: Is it possible for all pairs $(\boldsymbol{x}_0, \boldsymbol{x}_{\rm e})$ to find such an input function $\boldsymbol{u}_{[0,t_{\rm e}]}$? If the answer is in the affirmative, the system Σ is called completely controllable.

Similarly, the observability of a system concerns the situation that the input $\boldsymbol{u}_{[0,t_{\rm e}]}$ and the output $\boldsymbol{y}_{[0,t_{\rm e}]}$ are measured over some finite time interval $[0, t_{\rm e}]$ and that one wants to reconstruct the unknown initial state $\boldsymbol{x}_0$. If this problem can be solved for any initial state $\boldsymbol{x}_0$ and any input $\boldsymbol{u}_{[0,t_{\rm e}]}$, the system Σ is said to be completely observable.

Both notions are summarised in the following definition:

Definition 7.1 (Controllability and observability)
The system Σ is said to be completely controllable *if for any pair $(\boldsymbol{x}_0, \boldsymbol{x}_{\rm e})$ there is an input function $\boldsymbol{u}_{[0,t_{\rm e}]}$ that moves the system state within an finite time interval $[0, t_{\rm e}]$ towards the final state $\boldsymbol{x}(t_{\rm e}) = \boldsymbol{x}_{\rm e}$. The system is said to be* completely observable *if it is possible to reconstruct the initial state $\boldsymbol{x}_0$ from the input $\boldsymbol{u}_{[0,t_{\rm e}]}$ and the output $\boldsymbol{y}_{[0,t_{\rm e}]}$.*

Well-known necessary and sufficient conditions for a system to be completely controllable or completely observable are summarised in the following theorem:

Lemma 7.1 (Controllability and observability tests)
The system Σ is completely controllable if and only if the following equivalent conditions are satisfied:

- *(KALMAN:) The controllability matrix*

$$\boldsymbol{S}_{\rm C} = \left(\boldsymbol{B}\ \ \boldsymbol{AB}\ \ \boldsymbol{A}^2\boldsymbol{B}\ ...\ \boldsymbol{A}^{n-1}\boldsymbol{B}\right) \tag{7.7}$$

  has full rank:

$$\text{rank } \boldsymbol{S}_{\rm C} = n. \tag{7.8}$$

- *(HAUTUS:) For all eigenvalues λ_i of the matrix $\boldsymbol{A}$ the following condition is satisfied:*

$$\text{rank } (\lambda_i \boldsymbol{I} - \boldsymbol{A}\ \ \boldsymbol{B}) = n, \quad i = 1, 2, ..., n. \tag{7.9}$$

The system Σ is completely observable if and only if the following equivalent conditions are satisfied:

- *(KALMAN:) The observability matrix*

$$\boldsymbol{S}_{\rm O} = \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{pmatrix} \tag{7.10}$$

  has full rank:

$$\text{rank } \boldsymbol{S}_{\rm O} = n. \tag{7.11}$$

- *(HAUTUS:) For all eigenvalues λ_i of the matrix $\boldsymbol{A}$ the following condition is satisfied:*

$$\text{rank } \begin{pmatrix} \lambda_i \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{pmatrix} = n, \quad i = 1, 2, ..., n. \tag{7.12}$$

Obviously, these conditions do not depend upon the choice of the time $t_{\rm e}$ and, thus, the properties of controllability and observability are valid for arbitrary $t_{\rm e} > 0$. As in the theorem above, the two equivalent conditions are often referred to with the name of their author. Since the controllability tests concern the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and the observability tests the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ one also says that the pairs $(\boldsymbol{A}, \boldsymbol{B})$ or $(\boldsymbol{A}, \boldsymbol{C})$ are completely controllable or completely observable, respectively, if the conditions of Lemma 7.1 are satisfied.

The criteria (7.9) and (7.12) can be used to associate the controllability and the observability property to the eigenvalues of the matrix $\boldsymbol{A}$. This is an important step if the system is not completely observable or not completely controllable and one wants to know which part of the system cannot be controlled or observed. If (7.9) or (7.12) is violated for an eigenvalue λ_i, this eigenvalue is said to be uncontrollable or unobservable. Furthermore, such eigenvalues are said to be *input decoupling zeros* or *output decoupling zeros* of the system, respectively.

The connection between uncontrollable or unobservable eigenvalues and input or output decoupling zeros makes clear that a structural analysis that finds uncontrollable or unobservable parts of a system simultaneously shows the existence of decoupling zeros. Note that the structural results will concern the existence of such eigenvalues and zeros, but cannot find the numerical value of them.

Duality. Controllability and observability are dual properties in the following sense. If one considers the system

$$\Sigma_{\rm D}: \begin{cases} \dot{\boldsymbol{x}}_{\rm D}(t) = \boldsymbol{A}^{\rm T}\boldsymbol{x}_{\rm D}(t) + \boldsymbol{C}^{\rm T}\boldsymbol{u}_{\rm D}(t), \quad \boldsymbol{x}_{\rm D}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}_{\rm D}(t) = \boldsymbol{B}^{\rm T}\boldsymbol{x}_{\rm D}(t) \end{cases} \tag{7.13}$$

and applies the controllability criteria to $(\boldsymbol{A}^{\rm T}, \boldsymbol{C}^{\rm T})$ then one checks the observability of the original pair $(\boldsymbol{A}, \boldsymbol{C})$ and vice versa. $\Sigma_{\rm D}$ is said to be the dual system of Σ. Consequently, the following considerations can focus on the controllability property and derive the corresponding results on the observability by utilising this duality property.

A further important aspect of these properties is their invariance with respect to state transformations. If the state-space model (7.2) of Σ is formulated in terms of a new state vector

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{T}^{-1}\boldsymbol{x}(t) \tag{7.14}$$

with a regular transformation matrix $\boldsymbol{T}$, then the new representation of Σ satisfies the controllability and observability conditions if and only if the original model has satisfied them.

Example 7.2 *System with an output decoupling zero*

The system

$$\Sigma: \begin{cases} \dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y(t) = (1 \;\; 0 \;\; 0)\, \boldsymbol{x}(t) \end{cases}$$

with the scalar input $u(t)$ and the scalar output $y(t)$ satisfies the Kalman controllability criterion (7.8)

$$\text{rank}\, \boldsymbol{S}_{\text{C}} = \text{rank} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix} = 3,$$

but does not pass the observability test (7.11):

$$\text{rank}\ \boldsymbol{S}_{\text{O}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} = 2 < 3.$$

The Hautus criterion (7.12) reveals that the eigenvalue $\lambda_1 = 1$ is not observable and, hence, is an output decoupling zero.

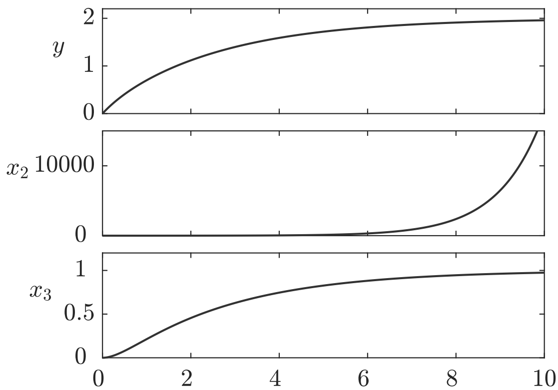

Fig. 7.4: Step response and behaviour of the state variables $x_2(t)$ and $x_3(t)$

A consequence of these properties can be seen in Fig. 7.4, which shows the step response in the top subplot and the behaviour of the state variables $x_2(t)$ and $x_3(t)$ in the two other plots. The step input stimulates the exponentially increasing behaviour of $x_2(t)$, which has the analytical expression

$$\begin{aligned} x_2(t) &= (0\ \ 1\ \ 0)\, \boldsymbol{A}^{-1} \text{e}^{\boldsymbol{A}t} \boldsymbol{b} - (0\ \ 1\ \ 0) \boldsymbol{A}^{-1} \boldsymbol{b} \\ &= 3 + 0.8\text{e}^{\,t} + 2.218\text{e}^{\,-0.362t} - 0.0181\text{e}^{\,-2.618t} \end{aligned}$$

with the term $\text{e}^{\,t}$ including the eigenvalue $\lambda_1 = 1$. This term does not appear in the output

$$\begin{aligned} y(t) &= \boldsymbol{c}^{\text{T}} \boldsymbol{A}^{-1} \text{e}^{\boldsymbol{A}t} \boldsymbol{b} - \boldsymbol{c}^{\text{T}} \boldsymbol{A}^{-1} \boldsymbol{b} \\ &= 2 - 1.896\text{e}^{\,-0.362t} - 0.1056\text{e}^{\,-2.618t} \overset{t \to \infty}{\longrightarrow} 2 \end{aligned}$$

that converges to the static reinforcement $k_{\text{s}} = 2$. It is in line with the notion of a zero to say that Σ has an output decoupling zero at $\lambda_1 = 1$, because the internal movement of the system along the function $\text{e}^{\,\lambda_1 t}$ is blocked when considering the transition from the state vector $\boldsymbol{x}(t)$ towards the output $y(t)$. □

7.2.3 Canonical structure of linear systems

If a system is not completely controllable, it can be decomposed into a controllable subsystem and a remaining subsystem that does not have any input. This decomposition is arranged by a state transformation (7.14) with an appropriate matrix $\boldsymbol{T}$ leading to the model

$$\Sigma : \begin{cases} \begin{pmatrix} \dot{\tilde{\boldsymbol{x}}}_1(t) \\ \dot{\tilde{\boldsymbol{x}}}_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \boldsymbol{O} & \tilde{\boldsymbol{A}}_{22} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{x}}_1(t) \\ \tilde{\boldsymbol{x}}_2(t) \end{pmatrix} + \begin{pmatrix} \tilde{\boldsymbol{B}}_1 \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = (\tilde{\boldsymbol{C}}_1 \quad \tilde{\boldsymbol{C}}_2) \begin{pmatrix} \tilde{\boldsymbol{x}}_1(t) \\ \tilde{\boldsymbol{x}}_2(t) \end{pmatrix} \end{cases} \tag{7.15}$$

with a completely controllable pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{B}}_1)$, whereas the input $\boldsymbol{u}(t)$ affects only the vector $\tilde{\boldsymbol{x}}_1(t)$, but not the vector $\tilde{\boldsymbol{x}}_2(t)$. This property is visible in the block diagram of Fig. 7.5.

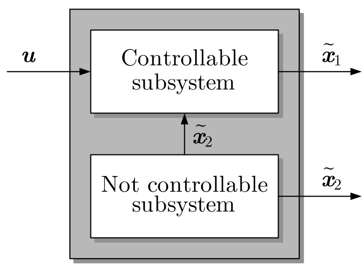

Fig. 7.5: Decomposition of a linear system with respect to its controllability

A similar decomposition can be made with respect to the observability of the system. The combination of both decompositions is called the *Kalman decomposition*. As shown in the large block in Fig. 7.6, the system Σ is represented by four subsystems

$$\Sigma : \begin{cases} \begin{pmatrix} \dot{\tilde{\boldsymbol{x}}}_1(t) \\ \dot{\tilde{\boldsymbol{x}}}_2(t) \\ \dot{\tilde{\boldsymbol{x}}}_3(t) \\ \dot{\tilde{\boldsymbol{x}}}_4(t) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} & \tilde{\boldsymbol{A}}_{13} & \tilde{\boldsymbol{A}}_{14} \\ \boldsymbol{O} & \tilde{\boldsymbol{A}}_{22} & \boldsymbol{O} & \tilde{\boldsymbol{A}}_{24} \\ \boldsymbol{O} & \boldsymbol{O} & \tilde{\boldsymbol{A}}_{33} & \tilde{\boldsymbol{A}}_{34} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \tilde{\boldsymbol{A}}_{44} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{x}}_1(t) \\ \tilde{\boldsymbol{x}}_2(t) \\ \tilde{\boldsymbol{x}}_3(t) \\ \tilde{\boldsymbol{x}}_4(t) \end{pmatrix} + \begin{pmatrix} \tilde{\boldsymbol{B}}_1 \\ \tilde{\boldsymbol{B}}_2 \\ \boldsymbol{O} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = (\boldsymbol{O} \quad \tilde{\boldsymbol{C}}_2 \quad \boldsymbol{O} \quad \tilde{\boldsymbol{C}}_4) \begin{pmatrix} \tilde{\boldsymbol{x}}_1(t) \\ \tilde{\boldsymbol{x}}_2(t) \\ \tilde{\boldsymbol{x}}_3(t) \\ \tilde{\boldsymbol{x}}_4(t) \end{pmatrix} \end{cases} \tag{7.16}$$

with the following properties:

- The pairs $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{B}}_1)$ and $(\tilde{\boldsymbol{A}}_{22}, \tilde{\boldsymbol{B}}_2)$ are completely controllable.
- The pairs $(\tilde{\boldsymbol{A}}_{22}, \tilde{\boldsymbol{C}}_2)$ and $(\tilde{\boldsymbol{A}}_{44}, \tilde{\boldsymbol{C}}_4)$ are completely observable.

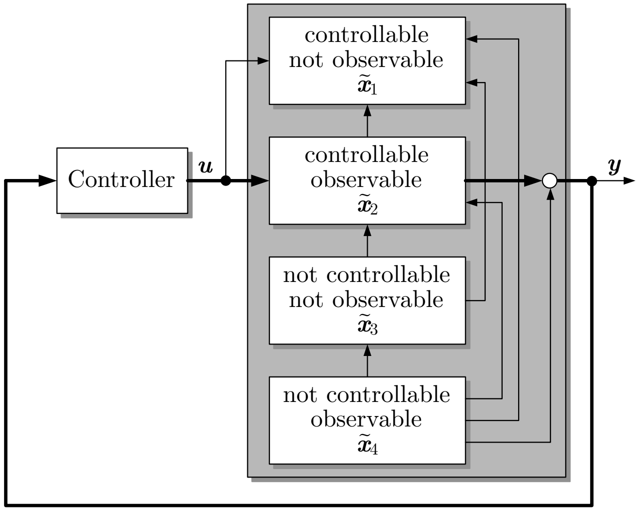

Fig. 7.6: Kalman decomposition and feedback controller

The other pairs are obviously not controllable or not observable. The transformed model (7.16) represents the *canonical structure* of linear systems. Every system can be decomposed into these four subsystems. However, in many applications, only one or two subsystems actually appear with a non-trivial state vector $(\dim \tilde{\boldsymbol{x}}_i > 0)$.

The decomposition clearly shows that only the controllable and observable subsystem lies in the feedback loop, which is closed by a controller coupling the signals $\boldsymbol{y}(t)$ and $\boldsymbol{u}(t)$. In particular, any feedback controller can only change the eigenvalues of the matrix $\tilde{\boldsymbol{A}}_{22}$, whereas for all closed-loop systems the eigenvalues of the other three diagonal blocks $\tilde{\boldsymbol{A}}_{11}$, $\tilde{\boldsymbol{A}}_{33}$ and $\tilde{\boldsymbol{A}}_{44}$ remain the same. One says that these eigenvalues are fixed, as the following section will explain.

7.2.4 Fixed eigenvalues

The eigenvalues $\lambda_i\{\boldsymbol{A}\}$, $(i = 1, 2, ..., n)$ appearing in the model (7.2) are the elements of the spectrum of $\boldsymbol{A}$

$$\sigma(\boldsymbol{A}) = \Big\{\lambda_1\{\boldsymbol{A}\}, \lambda_2\{\boldsymbol{A}\}, ..., \lambda_n\{\boldsymbol{A}\}\Big\}$$

and can be classified with respect to their controllability and observability properties. They are said to be controllable and observable if they satisfy the rank conditions (7.9) and (7.12). Note that now the controllability and observability properties are associated with any single eigenvalue, whereas in Definition 7.1 these properties have been defined for the system Σ as a whole. The set of all controllable and observable eigenvalues is

$$\sigma_{\rm CO}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) = \left\{\lambda \in \sigma(\boldsymbol{A}) \;\middle|\; \text{rank}\,(\lambda\boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{B}) = n \text{ and rank} \begin{pmatrix} \lambda\boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{pmatrix} = n \right\}.$$

The remaining eigenvalues are said to be *fixed*:[1]

$$\sigma_{\rm fix}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) = \left\{ \lambda \in \sigma(\boldsymbol{A}) \,\middle|\, \text{rank}\,(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{B}) < n \text{ or rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{pmatrix} < n \right\}.$$

The spectrum of $\boldsymbol{A}$ can be partitioned into disjoint sets as

$$\sigma(\boldsymbol{A}) = \sigma_{\rm CO}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) \cup \sigma_{\rm fix}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}).$$

The important property of the fixed eigenvalues is that no controller $\boldsymbol{K}_{\rm y}$ can change these eigenvalues. This fact is represented by the relation

$$\sigma_{\rm fix}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) = \bigcap_{\boldsymbol{K}_{\rm y} \in {\rm IR}^{m\times p}} \sigma(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C}), \tag{7.17}$$

where the intersection is determined with respect to any feedback matrix that may be applied to a system with m inputs and p outputs. As usual in multivariable control, it is assumed that any output $y_i(t)$, $(i = 1, 2, ..., p)$ can be connected by a controller with any input $u_j(t)$, $(j = 1, 2, ..., m)$, which means that the controller is unconstrained.

The important aspect of the controllability and observability properties introduced in Definition 7.1 is the fact that all controllable and observable eigenvalues can be changed by some feedback $\boldsymbol{K}_{\rm y}$. These eigenvalues appear in the spectrum $\sigma(\boldsymbol{A})$ of the plant, but, for any appropriate controller matrix, not in the spectrum $\sigma(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C})$ of the closed-loop system and, hence, not in $\sigma_{\rm fix}$. With other words, controllable and observable eigenvalues are "changeable" by feedback control. Correspondingly, if a system is completely controllable and completely observable then for any $\lambda \in \mathbb{C}$ there exists a feedback matrix $\boldsymbol{K}_{\rm y}$ such that

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C}) \neq 0$$

holds. By using an appropriate feedback matrix $\boldsymbol{K}_{\rm y}$, one can "avoid" every complex value λ as an eigenvalues of the closed-loop system. In particular, one can "avoid" the eigenvalue $\lambda = 0$ and make the closed-loop system matrix regular:

$$\exists \boldsymbol{K}_{\rm y}: \quad \det(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\rm y}\boldsymbol{C}) \neq 0. \tag{7.18}$$

For the sake of completeness, it should be mentioned that a static output feedback (7.3) is generally not capable of moving all changeable eigenvalues to arbitrarily given values. This aim can generally be reached if either the whole state vector is fed back by a state feedback

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t)$$

or if the controller has some dynamics. In particular, if the controller includes an observer for the current state $\boldsymbol{x}(t)$ and feeds the approximation $\hat{\boldsymbol{x}}(t)$ back

$$\boldsymbol{u}(t) = -\boldsymbol{K}\hat{\boldsymbol{x}}(t),$$

[1] In some publications, not the eigenvalues λ_i but the corresponding modes ${\rm e}^{\lambda_i t}$ of the linear system are said to be fixed.

it is possible to assign the eigenvalues of the closed-loop system predefined values. The following investigations only decide whether the eigenvalues of a system Σ are "changeable" or not, but it is not investigated how the eigenvalues can be shifted in a certain way.

Structurally constrained controllers. The investigations above can be generalised for decentralised controllers

$$\boldsymbol{u}(t) = -\boldsymbol{K}_{\mathrm{y}}\boldsymbol{y}(t) \quad \text{with} \quad \boldsymbol{K}_{\mathrm{y}} = \begin{pmatrix} k_{11} & & & \\ & k_{22} & & \\ & & \ddots & \\ & & & k_{NN} \end{pmatrix}, \tag{7.19}$$

for which many parameters of the controller are fixed to zero and only those in the main diagonal can be freely chosen. The structural analysis will show how these constraints manifest themselves in a reduced ability of the controller to change the eigenvalues of the closed-loop system. Instead of the fixed eigenvalues defined above for centralised controllers, a set of *decentralised fixed eigenvalues* occur that takes into consideration that the possible control matrices are constrained to belong to some set $\mathcal{K}_{\mathrm{y}}$. For decentralised controllers, $\mathcal{K}_{\mathrm{y}}$ includes only the diagonal matrices used in eqn. (7.19). Then the set of decentralised fixed eigenvalues (or decentralised fixed modes) is defined to be

$$\sigma_{\mathrm{dec}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) = \bigcap_{\boldsymbol{K}_{\mathrm{y}} \in \mathcal{K}_{\mathrm{y}}} \sigma(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\mathrm{y}}\boldsymbol{C}), \tag{7.20}$$

which is a superset of the set σ_{fix} defined above:

$$\sigma_{\mathrm{dec}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \supseteq \sigma_{\mathrm{fix}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}).$$

In the following investigations the set of fixed eigenvalues is considered with a complete (centralised) controller without mentioning, but the structural analysis will show in an intuitively clear manner how constraints on the feedback matrix affect the set of fixed eigenvalues.

7.3 Structural controllability and structural observability

7.3.1 Motivation

Controllability and observability are fundamental systems-theoretical notions that concern the couplings among the state variables and between the state and the actuators or the sensors, respectively. If one considers these properties from an abstract viewpoint, the following conjecture seems to be obvious, which refers to a graph-theoretical representation of the system that shows which signals are coupled:

> **Conjecture:** A dynamical system is completely controllable if in its structure graph all state vertices x_i, $(i = 1, 2, ..., n)$ are reachable from at least one input vertex u_i, $(i = 1, 2, ..., m)$ to facilitate an appropriate control action. The system is completely observable if there are signal paths from any of the state variables x_i to at least one of the output signals y_j, $(j = 1, 2, ..., p)$ each so that the measured output can be used to reconstruct the system state.

The following investigations should show that this conjecture is (nearly) true. They consider a structural description of the system Σ as a digraph, which distinguishes only between present and absent couplings among the signals of the state-space model of Σ. Graph-theoretical methods will lead to necessary and sufficient conditions for the newly introduced properties of structural controllability and structural observability, which include, besides the suspected reachability properties, an additional rank condition for the structure matrices of Σ.

7.3.2 Structure graph of linear dynamical systems

The structure of the dynamical system

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t), \end{cases} \tag{7.21}$$

is determined by the position of the non-vanishing elements of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. As explained in detail in Section 5.3.1, one classifies the elements of these matrices as those that are non-zero for (almost) all meaningful parameter values of the system and those that are fixed to zero. Accordingly, these elements are replaced by an $*$ or 0 and the resulting matrices $[\boldsymbol{A}]$, $[\boldsymbol{B}]$ and $[\boldsymbol{C}]$ are said to be the corresponding structure matrices. For example, a reactor with the parameters F, V_1, V_2 denoting a flow and two volumes has the matrices

$$\boldsymbol{A} = \begin{pmatrix} -\frac{F}{V_1} & 0 \\ \frac{F}{V_2} & -\frac{F}{V_2} \end{pmatrix} \quad \longrightarrow \quad [\boldsymbol{A}] = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

It is important to notice that the structure matrix can be set up without knowing the precise numerical values of the parameters involved.

In order to distinguish between the numerical matrices appearing in a model and the structure matrices used in the analysis, the structure matrices are denoted by $\boldsymbol{S}$ with the corresponding indices: $\boldsymbol{S_A}$, $\boldsymbol{S_B}$ and $\boldsymbol{S_C}$. In literature, the representation of a system by these structure matrices is said to be a *structured system*, but this notion is not used here, because the numerical matrices have the same zero pattern as the structure matrices and are, thus, structured as well.

Since any structure matrix specifies a set of structurally equivalent matrices, the step from the system model (7.21) towards the structure matrices $\boldsymbol{S_A}$, $\boldsymbol{S_B}$ and $\boldsymbol{S_C}$ means to leave the investigations of a unique dynamical system Σ and to consider a set $\mathcal{S}$ of linear systems with the same structure. This set is represented by

$$\mathcal{S} = \{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) : [\boldsymbol{A}] = \boldsymbol{S_A},\ [\boldsymbol{B}] = \boldsymbol{S_B},\ [\boldsymbol{C}] = \boldsymbol{S_C}\}. \tag{7.22}$$

If a set $\mathcal{S}$ of systems should be defined by given structure matrices, the expression

$$\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B}, \boldsymbol{S_C})$$

is used. Since in many applications one sets up a state-space model with the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ and derives the class of systems from them, the set $\mathcal{S}$ is often defined as

$$\mathcal{S} = ([\boldsymbol{A}], [\boldsymbol{B}], [\boldsymbol{C}]).$$

Then $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are usually not matrices with numerical elements, but include analytical expressions in terms of the system parameters. With the relation

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathcal{S}$$

one means that a triple of specified matrices is considered that belongs to the set $\mathcal{S}$. All results of the following investigations have to hold true for such sets of systems.

Graph-theoretical interpretation of the system structure. An explanatory diagram of the class (7.22) of linear systems is obtained by drawing a directed graph

$$\underline{\text{Structure graph:}} \quad \overrightarrow{\mathcal{G}_\mathcal{S}} = (\mathcal{V}, \mathcal{E}) \tag{7.23}$$

with

- $\mathcal{V}$ – set of vertices representing all input signals u_i, all state variables x_i and all output signals y_i,
- $\mathcal{E}$ – set of edges that connect vertices if the corresponding element of the structure matrix $\boldsymbol{S_A}$, $\boldsymbol{S_B}$ or $\boldsymbol{S_C}$, respectively, is an asterisk.

As the names of the signals involved are used as the names of the vertices, the set $\mathcal{V}$ is composed of three sets

$$\mathcal{V} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}.$$

Correspondingly, one speaks of input vertices $u_i \in \mathcal{U}$, state vertices $x_i \in \mathcal{X}$ and output vertices $y_i \in \mathcal{Y}$. u_i is associated with the i–th column of $\boldsymbol{S_B}$, y_i with the i–th row of $\boldsymbol{S_C}$ and x_i with the i–th columns of $\boldsymbol{S_A}$ and $\boldsymbol{S_C}$ and with the i–th row of $\boldsymbol{S_A}$ and $\boldsymbol{S_B}$.

The adjacency matrix $\boldsymbol{Q}_\mathcal{S}$ of the structure graph $\overrightarrow{\mathcal{G}_\mathcal{S}}$ is given by

$$\boldsymbol{Q}_\mathcal{S} = \begin{pmatrix} \boldsymbol{S_A} & \boldsymbol{S_B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{S_C} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{matrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{matrix}$$

for the order of the vertices marked on the right-hand side. To remember the construction of this matrix, write the state-space model (7.21) in the form

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{C} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{pmatrix}. \tag{7.24}$$

The position of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ is the same as the position of the corresponding structure matrices in the adjacency matrix $\boldsymbol{Q}_\mathcal{S}$. The middle row, which now means $\boldsymbol{u}(t) = \boldsymbol{0}$ will be replaced by a controller in the later investigations (cf. eqn. (7.28)). The graph $\overrightarrow{\mathcal{G}_\mathcal{S}}$ has $N = n + m + p$ vertices and, thus, the matrix $\boldsymbol{Q}_\mathcal{S}$ the dimension $(N \times N)$.

For drawing a structure graph, memorise the following rules:

- There is a directed edge $(u_j \to x_i)$ if the ij–th element of $\boldsymbol{S_B}$ is an $*$.
- There is a directed edge $(x_j \to x_i)$ if the ij–th element of $\boldsymbol{S_A}$ is an $*$.
- There is a directed edge $(x_j \to y_i)$ if the ij–th element of $\boldsymbol{S_C}$ is an $*$.

The graph $\overrightarrow{\mathcal{G}_{\mathcal{S}}}$ shows the signal couplings that may appear in all systems that belong to the set $\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B}, \boldsymbol{S_C})$. For the controllability analysis, only that part of $\overrightarrow{\mathcal{G}_{\mathcal{S}}}$ has to be considered that concerns the input and the state vertices. This subgraph $\overrightarrow{\mathcal{G}_{\mathrm{C}}}$ with the vertex set $\mathcal{U} \cup \mathcal{X}$ has the adjacency matrix

$$\boldsymbol{Q}_{\mathrm{C}} = \begin{pmatrix} \boldsymbol{S_A} & \boldsymbol{S_B} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}$$

with the index “C” indicating the controllability analysis. For the observability analysis only the relations among the state vertices and the output vertices have to be considered, which are represented by the subgraph $\overrightarrow{\mathcal{G}_{\mathrm{O}}}$ of $\overrightarrow{\mathcal{G}_{\mathcal{S}}}$, where the index “O” is a hint to the observability analysis. This graph has the vertex set $\mathcal{X} \cup \mathcal{Y}$ and the adjacency matrix

$$\boldsymbol{Q}_{\mathrm{O}} = \begin{pmatrix} \boldsymbol{S_A} & \boldsymbol{O} \\ \boldsymbol{S_C} & \boldsymbol{O} \end{pmatrix}.$$

Example 7.3 *Structure graph of a third-order system*

Consider the system with the state-space model

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \\ 0 & b_{32} \end{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \begin{pmatrix} 0 & c_{12} & 0 \\ 0 & 0 & c_{23} \end{pmatrix} \boldsymbol{x}(t). \end{cases} \tag{7.25}$$

The model distinguishes between matrix elements that depend upon system parameters and those that are known to vanish. Therefore, one can read the structure matrices from this model as follows:

$$\boldsymbol{S_A} = \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \boldsymbol{S_B} = \begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & * \end{pmatrix}, \quad \boldsymbol{S_C} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{7.26}$$

With the adjacency matrix

$$\boldsymbol{Q}_{\mathcal{S}} = \left(\begin{array}{ccc|cc|cc} \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot \\ * & * & * & * & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & \cdot & * & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

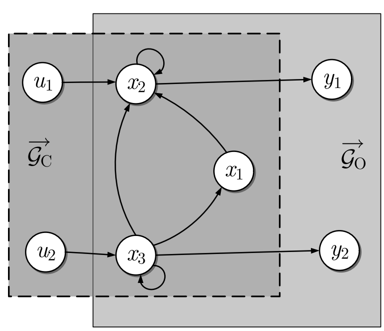

Fig. 7.7: Structure graph of the example system class

the graph $\overrightarrow{\mathcal{G}_S}$ in Fig. 7.7 is drawn. As it seems to be rather complicated to set up an adjacency matrix with so many zeros (symbolised by dots), one can use the rules given above to draw the graph directly. The controllability test below will only use the subgraph $\overrightarrow{\mathcal{G}_\mathrm{C}}$ and the observability tests the subgraph $\overrightarrow{\mathcal{G}_\mathrm{O}}$. □

7.3.3 Structure graph of closed-loop systems

If the system (7.21) is combined with an output feedback (7.3)

$$C: \quad \boldsymbol{u}(t) = -\boldsymbol{K}_\mathrm{y}\boldsymbol{y}(t)$$

the structure graph gets additional edges from all output vertices towards all input vertices. Since the feedback matrix $\boldsymbol{K}_\mathrm{y}$ may have any parameters, its $(m \times p)$–structure matrix $\boldsymbol{S_K}$

$$\text{Centralised controller:} \quad \boldsymbol{S_K} = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{pmatrix}. \tag{7.27}$$

has only $*$ elements. The extension of the model by the feedback term in

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{K}_\mathrm{y} \\ \boldsymbol{C} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{pmatrix} \tag{7.28}$$

leads to the extended adjacency matrix

$$\boldsymbol{Q}_\mathrm{F} = \begin{pmatrix} \boldsymbol{S_A} & \boldsymbol{S_B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{S_K} \\ \boldsymbol{S_C} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{matrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{matrix}$$

and the new structure graph denoted by $\overrightarrow{\mathcal{G}_\mathrm{F}}$ with the index "F" for the considered class of feedback loops. Figure 7.8 depicts the corresponding extension of the structure graph of Fig. 7.7. It clearly shows the two parts representing the structure of the plant and of the controller.

An alternative representation of the class of feedback loops is obtained if the structure graph is set up for the model (7.4)

$$\overline{\Sigma}: \quad \dot{\boldsymbol{x}}(t) = \underbrace{(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_\mathrm{y}\boldsymbol{C})}_{\overline{\boldsymbol{A}}}\boldsymbol{x}(t)$$

of the closed-loop system, for which the structure graph is denoted by $\overrightarrow{\mathcal{G}_\mathrm{R}}$. This model does not explicitly represent the inputs and the output of the plant. The graph has only state vertices and the adjacency matrix results from the structure matrices of the components as

$$\boldsymbol{Q}_\mathrm{R} = \boldsymbol{S_A} + \boldsymbol{S_B}\boldsymbol{S_K}\boldsymbol{S_C}.$$

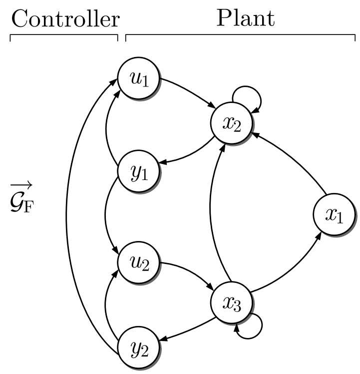

Fig. 7.8: Structure graph of an example with output feedback

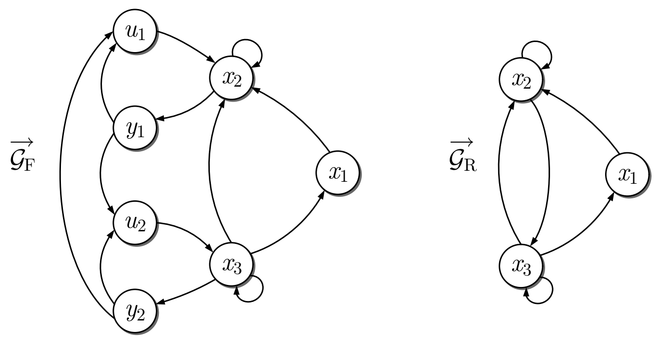

Fig. 7.9: Comparison of the two structure graphs

The two graphs of the feedback loop are compared in Fig. 7.9 for an example.

Structure graph for feedback loops with structurally constrained controllers. In the graph-theoretical setting, the notion of structurally constrained controllers is easy to understand, because the structural constraints fix some elements of the feedback matrix to zero. If one cannot implement a feedback between the output $y_j(t)$ and the input $u_i(t)$, the controller parameter k_{ij} is fixed to zero, which is represented by a zero in the ij–th position of the structure matrix $\boldsymbol{S_K}$. For decentralised controllers the structure matrix is

$$\text{Decentralised controller:} \quad \boldsymbol{S_K} = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}. \tag{7.29}$$

Accordingly, the structure graph $\overrightarrow{\mathcal{G}_F}$ of the restricted class of closed-loop systems includes only edges between the vertices y_i and u_i of the same channel i.

Example 7.4 *Third-order systems with decentralised controller*

Figure 7.9 compares the graphs $\overrightarrow{\mathcal{G}_{\mathrm{F}}}$ and $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$ for the system class considered in Example 7.3. Obviously, when going from $\overrightarrow{\mathcal{G}_{\mathrm{F}}}$ towards $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$, all paths from some state vertex (e. g. x_2) towards another state vertex (x_3) through output and input vertices (y_1 and u_2) are reduced to a direct edges $(x_2 \rightarrow x_3)$. The resulting graph $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$ is a concise representation of the class of feedback loops, in which all couplings are reduced to directed edges among the state vertices.

The following calculations, which use the arithmetic rules of p. 164, demonstrate how the adjacency matrix of the graph $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$ is obtained:

$$\begin{aligned} \boldsymbol{Q}_{\mathrm{R}} &= \boldsymbol{S_A} + \boldsymbol{S_B}\boldsymbol{S_K}\boldsymbol{S_C} \\ &= \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ 0 & * & * \end{pmatrix}. \end{aligned}$$

It is the adjacency matrix of the graph $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$ shown in Fig. 7.9.

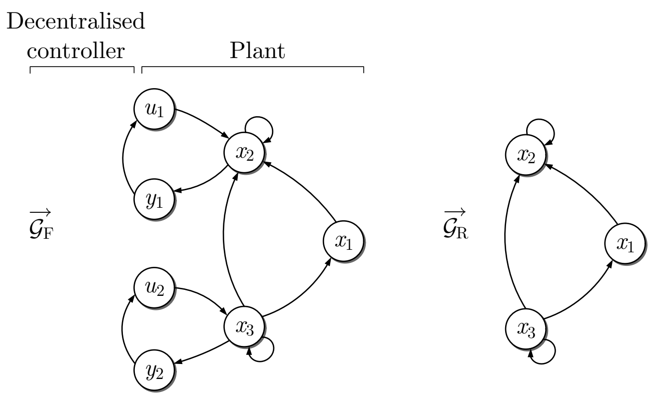

Fig. 7.10: Structure graphs of a feedback system with decentralised controllers

If the controller is restricted to be a decentralised controller, additional edges exist in the structure graph among the output vertices y_j and the input vertices u_i only for the same index: $i = j$. The structural constraints are visible in the reduction of the structure graph shown in Fig. 7.10 in comparison to the graph on the left-hand side of Fig. 7.9. The new adjacency matrix is obtained as follows:

$$\begin{aligned} \boldsymbol{Q}_{\mathrm{R}} &= \boldsymbol{S_A} + \boldsymbol{S_B}\boldsymbol{S_K}\boldsymbol{S_C} \\ &= \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ 0 & * & * \end{pmatrix}. \end{aligned}$$

The reduction of the freedom of the controller results in a graph $\overrightarrow{\mathcal{G}_{\mathrm{R}}}$ with fewer edges and, hence, different properties in comparison to a centralised controller. □

7.3.4 Reachability analysis of the structure graph

The conjecture stated at the beginning of this section on p. 247 can be expressed as a reachability property of the structure graph. For controllability, all state vertices should be reachable from at least one input vertex. As shown in Section 5.2.2, to raise the adjacency matrix to the power k means to follow paths in the graph with k edges. When applied to the adjacency matrix

$$\boldsymbol{Q}_{\mathrm{C}} = \begin{pmatrix} \boldsymbol{S_A} & \boldsymbol{S_B} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix},$$

which describes the connections among the input vertices and the state vertices, the result is

$$\boldsymbol{Q}_{\mathrm{C}}^k = \begin{pmatrix} \boldsymbol{S}_{\boldsymbol{A}}^k & \boldsymbol{S}_{\boldsymbol{A}}^{k-1}\boldsymbol{S_B} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}.$$

Since the conjecture refers to the reachability of the state vertices x_i, $(i = 1, 2, ..., n)$ from the input vertices u_i, $(i = 1, 2, ..., m)$, only the marked upper right part of $\boldsymbol{Q}_{\mathrm{C}}^k$ is important. Paths starting in an input vertex u_j and leading through $k-1$ state vertices to the state vertex x_i exist if the ij–th element of $\boldsymbol{S}_{\boldsymbol{A}}^{k-1}\boldsymbol{S_B}$ is non-zero. Consequently, the reachability of the state vertices from the input vertices are described by the *state reachability matrix* $\tilde{\boldsymbol{R}}_{\mathrm{xu}} = (\tilde{r}_{ij})$

$$\tilde{\boldsymbol{R}}_{\mathrm{xu}} = \sum_{k=0}^{n-1} \boldsymbol{S}_{\boldsymbol{A}}^k \boldsymbol{S_B} \tag{7.30}$$

with the following result:

$$\tilde{r}_{ij} \neq 0 \quad \Longleftrightarrow \quad \begin{array}{l}\text{the state vertex } x_i \text{ is reachable} \\ \text{from the input vertex } u_j.\end{array} \tag{7.31}$$

All state variables are reachable from at least one input vertex if the matrix $\tilde{\boldsymbol{R}}_{\mathrm{xu}}$ does not have any zero row. Then the class of systems $\mathcal{S}$ with the structure graph $\overrightarrow{\mathcal{G}_{\mathrm{C}}}$ under consideration is said to be *input connected* (or input-connectable or input-reachable).

If a system class $\mathcal{S}$ is not input connected its model can be transformed such that it gets the form (7.15) for which the structure graph of the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{B}}_1)$ is input connected. In the transformation relation (7.14) the matrix $\boldsymbol{T}$ is now a permutation matrix $\boldsymbol{P}$ that separates the reachable part of the state vertices from the vertices that are not reachable:

$$\boldsymbol{P}^{\mathrm{T}}(\boldsymbol{A}\ \ \boldsymbol{B}) \begin{pmatrix} \boldsymbol{P} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} = \left(\begin{array}{cc:c} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} & \tilde{\boldsymbol{B}}_1 \\ \boldsymbol{O} & \tilde{\boldsymbol{A}}_{22} & \boldsymbol{O} \end{array}\right). \tag{7.32}$$

Interpretation of the reachability for a discrete-time system. The reachability analysis has a nice interpretation for discrete-time systems

$$\Sigma_{\mathrm{d}} : \begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(k) + \boldsymbol{B}_{\mathrm{d}}\boldsymbol{u}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k). \end{cases} \tag{7.33}$$

It is well known that the controllability and observability tests stated in Lemma 7.1 are valid for discrete-time systems after the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have been replaced by the matrices $\boldsymbol{A}_{\rm d}$ and $\boldsymbol{B}_{\rm d}$. In the structure graph of $\Sigma_{\rm d}$ the matrices

$$\boldsymbol{S}_{\boldsymbol{B}_{\rm d}}, \quad \boldsymbol{S}_{\boldsymbol{A}_{\rm d}}\boldsymbol{S}_{\boldsymbol{B}_{\rm d}}, \quad \boldsymbol{S}^2_{\boldsymbol{A}_{\rm d}}\boldsymbol{S}_{\boldsymbol{B}_{\rm d}} \ \dots \ \boldsymbol{S}^k_{\boldsymbol{A}_{\rm d}}\boldsymbol{S}_{\boldsymbol{B}_{\rm d}}$$

show which state variables are influenced by the input $\boldsymbol{u}(0)$ at time k. Accordingly, if the ij–th element of $\boldsymbol{S}_{\boldsymbol{B}_{\rm d}}$ is an $*$, the state variable $x_i(1)$ is affected by the input $u_j(0)$ at time $k = 1$. If the ij–th element of $\boldsymbol{S}_{\boldsymbol{A}_{\rm d}}\boldsymbol{S}_{\boldsymbol{B}_{\rm d}}$ is non-zero, the state variable $x_i(2)$ is affected by $u_j(0)$ at time $k = 2$ etc.

For example, if the discrete-time system (7.33) has the same structure matrices as the system in Example 7.3, the structure graph in Fig. 7.7 reveals that the input $u_2(0)$ has a direct influence only on $x_3(1)$ but in the next time step also on $x_1(2)$ and $x_2(2)$, which is likewise shown by the second column of the matrices

$$\boldsymbol{S}_{\boldsymbol{B}_{\rm d}} = \begin{pmatrix} * & 0 \\ 0 & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_{\boldsymbol{A}_{\rm d}}\boldsymbol{S}_{\boldsymbol{B}_{\rm d}} = \begin{pmatrix} * & * \\ 0 & * \\ 0 & * \end{pmatrix}.$$

Output reachability. In analogy, the reachability of the output vertices from the state vertices can be checked by means of the *output reachability matrix* $\tilde{\boldsymbol{R}}_{\rm yx} = (\tilde{r}_{ij})$

$$\tilde{\boldsymbol{R}}_{\rm yx} = \sum_{k=0}^{n-1} \boldsymbol{S_C}\boldsymbol{S}^k_{\boldsymbol{A}}$$

with the following result:

$$\tilde{r}_{ij} \neq 0 \quad \Longleftrightarrow \quad \begin{array}{l}\text{the output vertex } y_i \text{ is reachable} \\ \text{from the state vertex } x_j.\end{array} \tag{7.34}$$

There is at least one output vertex reachable from any state vertex if the matrix $\tilde{\boldsymbol{R}}_{\rm yx}$ does not have any zero column. Then the class of systems $\mathcal{S}$ is said to be *output connected* (or output connectable or output-reachable).

7.3.5 Structural controllability of a class of linear systems

Since the structural analysis concerns a class $\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B}, \boldsymbol{S_C})$ of systems rather than a specific system $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, Definition 7.1 is not applicable and new notions of controllability and observability have to be introduced. The following definition extends the controllability notion to a class of systems such that the new structural property requires all signal couplings that are necessary for the complete controllability according to Definition 7.1. As the matrix pair $(\boldsymbol{A}, \boldsymbol{B})$ is relevant, the class of systems to be considered is

$$\mathcal{S}(\boldsymbol{S_A}, \boldsymbol{S_B}) = \{(\boldsymbol{A}, \boldsymbol{B}) : \ [\boldsymbol{A}] = \boldsymbol{S_A}, [\boldsymbol{B}] = \boldsymbol{S_B}\}. \tag{7.35}$$

Definition 7.2 (Structural controllability)
A class of systems $\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B})$ is said to be structurally controllable if there is a system $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{S}$ that is completely controllable.

In literature, the notion of structural controllability is sometimes abbreviated as s-controllability or one speaks of generic controllability. If the distinction between the structural controllability of a system class and the complete controllability of an individual system should be emphasised, the term "numerical controllability" is used for the latter notion.

The definition above has the following consequences:

Corollary 7.1 (Relation between controllability and structural controllability)
The structural controllability of the class $\mathcal{S}$ is a necessary condition for any system $\Sigma \in \mathcal{S}$ to be completely controllable.

The other way round, if the structural analysis introduced in the sequel finds out that the class $\mathcal{S}$ is not structurally controllable then all systems $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{S}$ are not completely controllable.

Σ is completely controllable.	$\Longrightarrow$	$\mathcal{S}$ with $\Sigma \in \mathcal{S}$ is structurally controllable.
All $\Sigma \in \mathcal{S}$ are not completely controllable.	$\Longleftarrow$	$\mathcal{S}$ is not structurally controllable.

At a first sight, one may doubt that the structural controllability does have a meaning for the individual systems $\Sigma \in \mathcal{S}$, because only a single controllable system Σ is required to call the infinite set $\mathcal{S}$ structurally controllable. However, these doubts are unjustified because with a single system Σ almost all systems in the set $\mathcal{S}$ are completely controllable. The controllability is a generic property. The parameters of linear systems, which belong to a structurally controllable set $\mathcal{S}$, but are not completely controllable, lie on a hypersurface in the parameter space of these systems (cf. Example 7.6).

Preliminary investigations. Before necessary and sufficient conditions for the structural controllability are derived, a hint to an important pitfall is appropriate. The notion of the structural rank introduced in Section 5.3.5 suggests to investigate the structure matrix $[\boldsymbol{S}_{\rm C}]$ resulting from the controllability matrix to get a test for structural controllability. If $\text{s\_rank}\,[\boldsymbol{S}_{\rm C}] = n$ holds, it seems to be evident that the inequality (5.58) implies that almost all systems of the class $\mathcal{S}$ considered satisfy the Kalman criterion (7.8). However, this conclusion is a fallacy, because the inequality (5.58) is valid only for matrices in which all $*$ elements have independent values. This independence assumption is not satisfied for the controllability matrix $\boldsymbol{S}_{\rm C}$ because the $n \cdot nm$ elements of $\boldsymbol{S}_{\rm C}$ can be traced back to the $n \cdot (n+m)$ elements of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. A full structural rank of $[\boldsymbol{S}_{\rm C}]$ does not imply that there are matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{S}_{\rm C}$ has full (numerical) rank and, in particular, it does not imply that for almost all matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ the controllability test (7.8) is satisfied.

Example 7.5 *An uncontrollable system with full structural rank*

Consider a third-order system that has the matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}. \tag{7.36}$$

The controllability matrix has the following properties:

$$\begin{aligned} \boldsymbol{S}_{\mathrm{C}} &= \begin{pmatrix} b & a_{11}b & (a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})b \\ 0 & a_{21}b & a_{11}a_{21}b \\ 0 & a_{31}b & a_{11}a_{31}b \end{pmatrix} \\ \det \boldsymbol{S}_{\mathrm{C}} &= b^3\,(a_{21}a_{11}a_{31} - a_{31}a_{11}a_{21}) = 0 \qquad (7.37) \\ \text{s\_rank}\,[\boldsymbol{S}_{\mathrm{C}}] &= \begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ 0 & * & \bullet \end{pmatrix} = 3. \qquad (7.38) \end{aligned}$$

However, there does not exist any set of entries of $(\boldsymbol{A}, \boldsymbol{B})$ in eqn. (7.36) such that the observability matrix has full rank. Hence, the set of systems

$$\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B}) \quad \text{with} \quad \boldsymbol{S_A} = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \boldsymbol{S_B} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \tag{7.39}$$

is not structurally controllable according to Definition 7.2.

Equations (7.37) and (7.38) seem to contradict the relation (5.58) on the structural rank. However, the elements of $\boldsymbol{S}_{\mathrm{C}}$ are not independent and, thus, the relation (5.58) is not valid for this matrix. □

Consequently, conditions for the structural controllability cannot be obtained by considering the structure matrix $[\boldsymbol{S}_{\mathrm{C}}]$, but have to be derived on other ways.

Necessary and sufficient conditions for structural controllability. The goal of this paragraph is to elaborate a test that for structure matrices $\boldsymbol{S_A}$ and $\boldsymbol{S_B}$ says whether the class $\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B})$ is structurally controllable. The first step is to prove that the conjecture presented in Section 7.3.1 provides a necessary condition (but not a sufficient condition).

According to Definition 7.2 the structural conditions have to ensure that there is at least one pair $(\boldsymbol{A}, \boldsymbol{B})$ with $\boldsymbol{A} \in \boldsymbol{S_A}$ and $\boldsymbol{B} \in \boldsymbol{S_B}$ that satisfies the Kalman criterion (7.8). With the relation (5.58) the rank of the controllability matrix $\boldsymbol{S}_{\mathrm{C}}$ satisfies the inequalities

$$\begin{aligned} \text{rank}\ \boldsymbol{S}_{\mathrm{C}} &\leq \text{s\_rank}\ \left([\boldsymbol{B}]\ \ [\boldsymbol{AB}]\ ...\ [\boldsymbol{A}^{n-1}\boldsymbol{B}]\right) \\ &\leq \text{s\_rank}\ \left([\boldsymbol{B}]\ \ [\boldsymbol{A}][\boldsymbol{B}]\ ...\ [\boldsymbol{A}]^{n-1}[\boldsymbol{B}]\right) \\ &\leq \text{s\_rank}\ \left(\boldsymbol{S_B}\ \ \boldsymbol{S_A}\boldsymbol{S_B}\ ...\ \boldsymbol{S_A}^{n-1}\boldsymbol{S_B}\right) \quad \text{for all}\ \ \boldsymbol{A} \in \boldsymbol{S_A}, \boldsymbol{B} \in \boldsymbol{S_B}. \end{aligned}$$

The second line results from eqn. (5.47). The third line is true, because a specific pair $(\boldsymbol{A}, \boldsymbol{B})$ may have less $*$ elements than the structure matrices defining the set $\mathcal{S}$. In order to make sure that the controllability matrix may have full rank, the relation

$$\text{s\_rank}\ \left(\boldsymbol{S_B}\ \ \boldsymbol{S_A}\boldsymbol{S_B}\ ...\ \boldsymbol{S_A}^{n-1}\boldsymbol{S_B}\right) = n$$

has to hold, which means that the matrix in the parantheses must not have a zero row. Equivalently, the sum of the matrices appearing in this matrix must not have a zero row, which can be posed as the requirement that the state reachability matrix (7.30)

$$\tilde{\boldsymbol{R}}_{\mathrm{xu}} = \sum_{k=0}^{n-1} \boldsymbol{S}_{\boldsymbol{A}}^{k} \boldsymbol{S}_{\boldsymbol{B}}$$

must not have a zero row. Hence, the structure graph $\overrightarrow{\mathcal{G}_{\mathrm{C}}}$ has to be input connected. In summary, to have an input connected graph is a necessary condition for the structural controllability of a class $\mathcal{S}$. To shorten the notation, if the graph is input connected then one also says that the class $\mathcal{S}$ is input connected.

The following counterexample shows that this property is not sufficient for the structural controllability.

Example 7.6 *Structural controllability analysis of two parallel integrators*

Consider the state equation of two parallel integrator systems

$$\Sigma : \quad \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{T_{\mathrm{I1}}} \\ \frac{1}{T_{\mathrm{I2}}} \end{pmatrix} u(t) \tag{7.40}$$

with the time constants T_{I1} and T_{I2}. This system belongs to the class $\mathcal{S}$ characterised by

$$\boldsymbol{S}_{\boldsymbol{A}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_{\boldsymbol{B}} = \begin{pmatrix} * \\ * \end{pmatrix}.$$

The structure graph $\overrightarrow{\mathcal{G}_{\mathrm{C}}}$ shown in Fig. 7.11 is input connected.

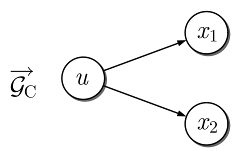

Fig. 7.11: Structure graph of two parallel integrators

However, in the class $\mathcal{S} = (\boldsymbol{S}_{\boldsymbol{A}}, \boldsymbol{S}_{\boldsymbol{B}})$ of all integrator systems (7.40) there is no system $(\boldsymbol{A}, \boldsymbol{B})$ that is completely controllable because the controllability matrix

$$\boldsymbol{S}_{\mathrm{C}} = \begin{pmatrix} \frac{1}{T_{\mathrm{I1}}} & 0 \\ \frac{1}{T_{\mathrm{I2}}} & 0 \end{pmatrix}$$

is singular for all parameters T_{I1} and T_{I2}. This example shows that the input connectivity of the structure graph is a necessary but not a sufficient condition for the system class to be structurally controllable. □

A second necessary condition should be derived now to exclude systems like the two-integrator system of Example 7.6. As the non-zero elements of $[\boldsymbol{A}]$ can be associated to any real number, each class $\mathcal{S}$ includes pairs $(\boldsymbol{A}, \boldsymbol{B})$ with a vanishing eigenvalue $\lambda_i = 0$, for which the Hautus criterion (7.9) implies the following inequalities:

$$\begin{aligned} \text{rank}\ (-\boldsymbol{A}\ \ \boldsymbol{B}) &\leq \text{s\_rank}\ ([\boldsymbol{A}]\ \ [\boldsymbol{B}]) \\ &\leq \text{s\_rank}\ (\boldsymbol{S_A}\ \ \boldsymbol{S_B}). \end{aligned}$$

The Hautus test can be satisfied by appropriate matrices $\boldsymbol{A} \in \boldsymbol{S_A}$ and $\boldsymbol{B} \in \boldsymbol{S_B}$ only if

$$\text{s\_rank}\ (\boldsymbol{S_A}\ \ \boldsymbol{S_B}) = n \tag{7.41}$$

holds. This condition is not satisfied by the integrator system considered in Example 7.6.

The following theorem states that the two necessary conditions together are also sufficient for the structural controllability of $\mathcal{S}$.

Theorem 7.1 (Structural controllability)
A class $\mathcal{S} = (\boldsymbol{S_A}, \boldsymbol{S_B})$ is structurally controllable if and only if the following two conditions are satisfied:

1. *$\mathcal{S}$ is input connected.*
2. *The condition (7.41) is satisfied.*

Outline of the proof. The necessity of the two conditions has already been proved by the investigations above. The sufficiency part is rather involved and should be abbreviated here by describing the proof steps.

The proof is accomplished by showing that almost all systems $\Sigma \in \mathcal{S}$ satisfy the Hautus criterion (7.9). Since the condition (7.41) ensures that eqn. (7.9) holds for $\lambda = 0$ for almost all $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{S}$, the crucial step is to prove that if $\mathcal{S}$ is input connected the Hautus criterion is satisfied for all $\lambda \neq 0$ for almost all $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{S}$. This proof is obtained as follows. For every choice of $\boldsymbol{A} \in \boldsymbol{S_A}$ and $\boldsymbol{B} \in \boldsymbol{S_B}$ the critical values of λ are the eigenvalues of $\boldsymbol{A}$, because for these values the matrix $\lambda\boldsymbol{I} - \boldsymbol{A}$ has a rank deficiency that has to be compensated within the matrix $(\lambda\boldsymbol{I} - \boldsymbol{A}\ \ \boldsymbol{B})$ by the matrix $\boldsymbol{B}$. A lengthy proof shows that if $\mathcal{S}$ is input connected, such a compensation is possible for almost all admissible matrices.

To illustrate this step, assume that the matrix $\boldsymbol{S_B}$ has no zero row. Then $\mathcal{S}$ is input connected for arbitrary $\boldsymbol{S_A}$. For any $\boldsymbol{A}$ the rank of the matrix $\lambda_i\boldsymbol{I} - \boldsymbol{A}$ with λ_i being an eigenvalue of $\boldsymbol{A}$ is less than n. However, for almost all matrices $\boldsymbol{B}$ the eigenvectors $\boldsymbol{v}_i$ belonging to λ_i lead to $\boldsymbol{B}\boldsymbol{v}_i \neq \boldsymbol{0}$ and the rank of $(\lambda_i\boldsymbol{I} - \boldsymbol{A}\ \ \boldsymbol{B})$ is equal to n as required. □