3

Models and Structure of Interconnected Systems

3.1 SUBSYSTEM AND OVERALL SYSTEM MODELS

In this section, the models of interconnected systems are summarized for later use. They are distinguished by the degree to which they reflect the internal structure of the overall system.

Unstructured model

From a global point of view, the plant is a dynamical system with m-dimensional input vector $\mathbf{u}$ and r-dimensional output vector $\mathbf{y}$ (Figure 3.1(a)). Its state space representation has the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{3.1.1}$$

where $\mathbf{x}$ denotes the n-dimensional state vector of the overall system. Since time-invariant systems will be considered, the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have constant elements and are of appropriate dimensions. The model (3.1.1) is well known from multivariable system theory, but is of minor importance for large-scale systems because it says nothing about the subsystems of the overall system.

I/O-oriented model

For decentralized control, the sensors and actuators are grouped to m_i- or r_i-dimensional vectors $\mathbf{u}_i$ and $\mathbf{y}_i$ ($i = 1, \ldots, N$), where the ith control station has access to $\mathbf{y}_i$ and determines $\mathbf{u}_i$ (Figure 3.1(b)). That is, the overall system input and output is decomposed into subvectors $\mathbf{u} = (\mathbf{u}_1' \;\; \mathbf{u}_2' \;\; \ldots \;\; \mathbf{u}_N')'$ and $\mathbf{y} = (\mathbf{y}_1' \;\; \mathbf{y}_2' \;\; \ldots \;\; \mathbf{y}_N')'$. Instead of eqn (3.1.1) the

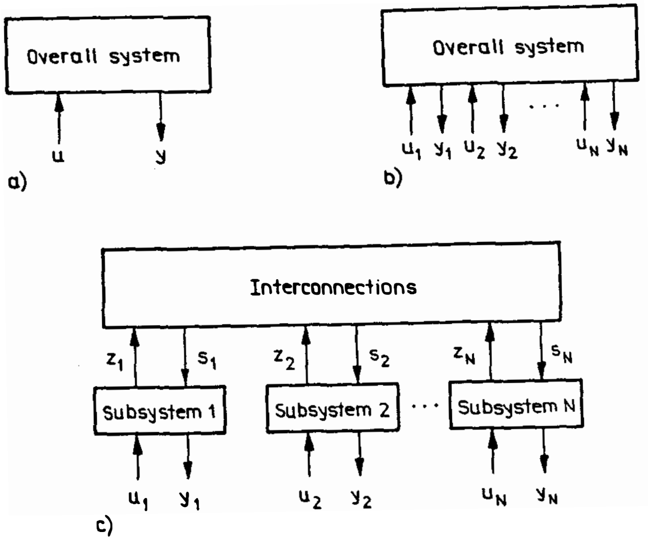

Figure 3.1 Structure of the models of interconnected systems: (a) unstructured model; (b) I/O oriented model; (c) interaction-oriented model

model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}_i(t) &= \mathbf{C}_{si}\mathbf{x}(t) + \sum_{j=1}^{N} \mathbf{D}_{ij}\mathbf{u}_j(t) \qquad (i = 1, \dots, N) \end{aligned} \tag{3.1.2}$$

is used which makes the structural constraints of the decentralized control perceptible. The matrices of eqn (3.1.2) can be obtained from (3.1.1) by decomposing **B**, **C** and **D** into submatrices, the dimensions of which are compatible with the dimensions of the vectors $\mathbf{u}_i$ and $\mathbf{y}_i$:

$$\mathbf{B} = (\mathbf{B}_{s1} \;\; \mathbf{B}_{s2} \;\; \dots \;\; B_{sN})$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{s1} \\ \mathbf{C}_{s2} \\ \vdots \\ \mathbf{C}_{sN} \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \dots & \mathbf{D}_{1N} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \dots & \mathbf{D}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{D}_{N1} & \mathbf{D}_{N2} & \dots & \mathbf{D}_{NN} \end{pmatrix}. \tag{3.1.3}$$

The model (3.1.2) exhibits the structure of the inputs and outputs but does not show how the overall system dynamics depends on the subsystems as the next form of the model will do.

Interaction-oriented model

Many large-scale systems emerge as a result of interactions between different subsystems. These couplings can have the nature of energy, material, or information flows. They are represented by signals $\mathbf{s}_i$ and $\mathbf{z}_i$ through which the ith subsystem interacts with other subsystems (Figure 3.1(c)). These additional input and output signals of the subsystems are internal signals of the overall system.

Since every subsystem represents a dynamical system of its own, it can be described by a state space model

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) + \mathbf{E}_i\mathbf{s}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0}\\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) + \mathbf{D}_i\mathbf{u}_i(t) + \mathbf{F}_i\mathbf{s}_i(t)\\ \mathbf{z}_i(t) &= \mathbf{C}_{zi}\mathbf{x}_i(t) + \mathbf{D}_{zi}\mathbf{u}_i(t) + \mathbf{F}_{zi}\mathbf{s}_i(t)\end{aligned} \tag{3.1.4}$$

where $\mathbf{x}_i$ is the n_i-dimensional state vector of the ith subsystem. Eqn (3.1.4) will be referred to as the ith subsystem. If the interactions between the subsystems are neglected ($\mathbf{s}_i(t) = \mathbf{0}$), eqn (3.1.4) yields the model of the isolated subsystem

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0}\\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) + \mathbf{D}_i\mathbf{u}_i(t).\end{aligned} \tag{3.1.5}$$

The interconnections of the subsystems (3.1.4) are described by

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{3.1.6}$$

where the vectors $\mathbf{s}$ and $\mathbf{z}$ of dimension m_s or r_z, respectively, consist of the interconnection inputs $\mathbf{s}_i$ and outputs $\mathbf{z}_i$ of the subsystems with dimensions m_{si} and r_{zi}: $\mathbf{s} = (\mathbf{s}_1' \; \mathbf{s}_2' \; \dots \; \mathbf{s}_N')'$, $\mathbf{z} = (\mathbf{z}_1' \; \dots \; \mathbf{z}_N')'$. The interconnection relation can be represented by the algebraic equation (3.1.6) if all the dynamical elements of the system are considered as part of some subsystem. The model (3.1.4) and (3.1.6) makes clear which subsystems comprise the whole system and which interactions exist among these subsystems.

Relation between the Unstructured Model and the Interaction-oriented Model

A representation of the overall system matrices **A**, **B**, **C** and **D** in terms of the subsystem matrices $\mathbf{A}_i$, $\mathbf{B}_i$, ... and the interconnection matrix **L** can be formulated as follows. Writing the subsystem equations (3.1.4)

($i = 1, \ldots, N$) one below the other leads to

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \text{diag } \mathbf{A}_i\mathbf{x}(t) + \text{diag } \mathbf{B}_i\mathbf{u}(t) + \text{diag } \mathbf{E}_i\mathbf{s}(t)\\ \mathbf{y}(t) &= \text{diag } \mathbf{C}_i\mathbf{x}(t) + \text{diag } \mathbf{D}_i\mathbf{u}(t) + \text{diag } \mathbf{F}_i\mathbf{s}(t)\\ \mathbf{z}(t) &= \text{diag } \mathbf{C}_{zi}\mathbf{x}(t) + \text{diag } \mathbf{D}_{zi}\mathbf{u}(t) + \text{diag } \mathbf{F}_{zi}\mathbf{s}(t)\end{aligned} \tag{3.1.7}$$

and $\mathbf{x}(0) = \mathbf{x}_0$ where

$$\mathbf{x} = (\mathbf{x}_1' \;\; \mathbf{x}_2' \;\; \ldots \;\; \mathbf{x}_N')' \tag{3.1.8}$$

and $\mathbf{u} = (\mathbf{u}_1' \;\ldots\; \mathbf{u}_N')'$ hold; diag $\mathbf{A}_i$ stands for a block-diagonal matrix with the diagonal blocks $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$. Eqns (3.1.6) and (3.1.7) yield

$$\begin{pmatrix}\mathbf{I} & \mathbf{0} & -\text{diag } \mathbf{E}_i\mathbf{L}\\ \mathbf{0} & \mathbf{I} & -\text{diag } \mathbf{F}_i\mathbf{L}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L}\end{pmatrix}\begin{pmatrix}\dot{\mathbf{x}}\\ \mathbf{y}\\ \mathbf{z}\end{pmatrix} = \begin{pmatrix}\text{diag } \mathbf{A}_i\\ \text{diag } \mathbf{C}_i\\ \text{diag } \mathbf{C}_{zi}\end{pmatrix}\mathbf{x} + \begin{pmatrix}\text{diag } \mathbf{B}_i\\ \text{diag } \mathbf{D}_i\\ \text{diag } \mathbf{D}_{zi}\end{pmatrix}\mathbf{u}. \tag{3.1.9}$$

The matrix on the left-hand side of eqn (3.1.9) is invertible if and only if

$$\det(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L}) \neq 0 \tag{3.1.10}$$

holds. If so, a model of the form (3.1.1) can be derived from eqn (3.1.9) where

$$\begin{aligned}\mathbf{A} &= \text{diag } \mathbf{A}_i + \text{diag } \mathbf{E}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{C}_{zi}\\ \mathbf{B} &= \text{diag } \mathbf{B}_i + \text{diag } \mathbf{E}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{D}_{zi}\\ \mathbf{C} &= \text{diag } \mathbf{C}_i + \text{diag } \mathbf{F}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{C}_{zi}\\ \mathbf{D} &= \text{diag } \mathbf{D}_i + \text{diag } \mathbf{F}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{D}_{zi}\end{aligned} \tag{3.1.11}$$

hold. Eqn (3.1.11) shows how the subsystem and interconnection parameters combine with the overall system parameters. These relations are easier to understand under the reasonable assumption that the subsystem models (3.1.4) have no direct throughput of $\mathbf{u}_i$ and $\mathbf{s}_i$ towards $\mathbf{z}_i$ and $\mathbf{y}_i$, that is

$$\mathbf{D}_i = \mathbf{0} \qquad \mathbf{F}_i = \mathbf{0} \qquad \mathbf{D}_{zi} = \mathbf{0} \qquad \mathbf{F}_{zi} = \mathbf{0} \tag{3.1.12}$$

($i = 1, \ldots, N$) hold. Then, after partitioning the interconnection matrix $\mathbf{L}$ in (3.1.5) according to the structure of $\mathbf{s}$ and $\mathbf{z}$

$$\mathbf{L} = \begin{pmatrix}\mathbf{L}_{11} & \mathbf{L}_{12} & \ldots & \mathbf{L}_{1N}\\ \mathbf{L}_{21} & \mathbf{L}_{22} & \ldots & \mathbf{L}_{2N}\\ \vdots & \vdots & & \vdots\\ \mathbf{L}_{N1} & \mathbf{L}_{N2} & \ldots & \mathbf{L}_{NN}\end{pmatrix} \tag{3.1.13}$$

eqn (3.1.11) has the simpler form

$$\begin{aligned}
&\mathbf{A} = (\mathbf{A}_{ij}) \quad \text{with} \quad \mathbf{A}_{ii} = \mathbf{A}_i + \mathbf{E}_i\mathbf{L}_{ii}\mathbf{C}_{zi} \\
&\qquad\qquad\qquad\quad\;\; \mathbf{A}_{ij} = \mathbf{E}_i\mathbf{L}_{ij}\mathbf{C}_{zj} \quad \text{for } i \neq j \\
&\mathbf{B} = \text{diag } \mathbf{B}_i \\
&\mathbf{C} = \text{diag } \mathbf{C}_i \\
&\mathbf{D} = \mathbf{0}.
\end{aligned} \tag{3.1.14}$$

Obviously, the subsystem matrices $\mathbf{A}_i$ occur as diagonal blocks of $\mathbf{A}$ whereas the interactions as described by $\mathbf{E}_i$, $\mathbf{C}_{zi}$ and $\mathbf{L}$ are parts of the non-diagonal blocks $\mathbf{A}_{ij}$ $(i \neq j)$. In particular, if, as often happens, the diagonal blocks of $\mathbf{L}$ vanish ($\mathbf{L}_{ii} = \mathbf{0}$ in eqn (3.1.13)) because the interconnection input $\mathbf{s}_i$ does not directly depend on the interconnection output $\mathbf{z}_i$ of the same subsystem, the diagonal blocks of $\mathbf{A}$ equal the subsystem matrices $\mathbf{A}_i$ ($\mathbf{A}_{ii} = \mathbf{A}_i$). $\mathbf{B}$ and $\mathbf{C}$ are block diagonal. If the subsystems have no direct throughput the same holds for the overall system ($\mathbf{D} = \mathbf{0}$).

Equation (3.1.14) says that under the assumption (3.1.12) the matrices $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$ of the I/O-oriented model (3.1.2) can be written as

$$\mathbf{B}_{si} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{B}_i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \qquad \mathbf{C}_{si} = (\mathbf{0} \;\ldots\; \mathbf{0} \;\; \mathbf{C}_i \;\; \mathbf{0} \;\ldots\; \mathbf{0}) \tag{3.1.15}$$

where only the ith block is non-vanishing. By using eqns (3.1.14) and (3.1.15) a further form of the overall system model is obtained

$$\begin{aligned}
&\dot{\mathbf{x}}_i(t) = \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\
&\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}_i(t) \qquad (i = 1, \ldots, N).
\end{aligned} \tag{3.1.16}$$

This model is said to have an *input–output decentralized form* (cf. Section 3.3). It will be used if the dependencies between the subsystem states $\mathbf{x}_i$ are investigated. In eqn (3.1.16) these dependencies are described by the matrices $\mathbf{A}_{ij}$.

In this context, the overall system matrix $\mathbf{A}$ is sometimes decomposed into

$$\mathbf{A}_{\mathrm{D}} = \text{diag } \mathbf{A}_{ii} \tag{3.1.17}$$

and

$$\mathbf{A}_\mathrm{C} = \mathbf{A} - \mathbf{A}_\mathrm{D} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1N} \\ \mathbf{A}_{21} & \mathbf{0} & \dots & \mathbf{A}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \dots & \mathbf{0} \end{pmatrix} \tag{3.1.18}$$

which represents the interaction relation.

Two further remarks have to be made concerning the relation of the models (3.1.1) and (3.1.4) and (3.1.6). First, (3.1.10) represents not merely a condition under which the overall system model (3.1.1) and (3.1.11) can be derived from (3.1.4) and (3.1.6), but it also ensures the existence of some model of the form (3.1.1) due to the uniqueness of the solution of (3.1.4) and (3.1.6).

Theorem 3.1

The equations (3.1.4) and (3.1.6) have a unique solution and can be represented in the form (3.1.1) if and only if the condition (3.1.10) is satisfied.

Proof

The sufficiency has been proved by constructing the model (3.1.1) and (3.1.11) from (3.1.4) and (3.1.6). In order to prove the necessity consider the last row

$$(\mathbf{I} - \mathrm{diag}\ \mathbf{F}_{zi}\mathbf{L})\mathbf{z} = \mathrm{diag}\ \mathbf{C}_{zi}\mathbf{x} + \mathrm{diag}\ \mathbf{D}_{zi}\mathbf{u} \tag{3.1.19}$$

of eqn (3.1.9). If the matrix $(\mathbf{I} - \mathrm{diag}\ \mathbf{F}_{zi}\mathbf{L})$ is singular, a zero row can be made to appear in this matrix by elementary row operations. Then, eqn (3.1.19) has the form

$$\begin{pmatrix} * \\ \mathbf{0} \end{pmatrix} \mathbf{z}(t) = \begin{pmatrix} * \\ \mathbf{a}' \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} * \\ \mathbf{b}' \end{pmatrix} \mathbf{u}(t)$$

where the asterisks denote arbitrary blocks and $\mathbf{a}'$ and $\mathbf{b}'$ row vectors. For $\mathbf{a}' \neq \mathbf{0}$ and $\mathbf{b}' \neq \mathbf{0}$ the last line

$$\mathbf{a}'\mathbf{x}(t) + \mathbf{b}'\mathbf{u}(t) = 0$$

states a linear dependence between $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Otherwise, $\mathbf{a}' = \mathbf{0}$ or $\mathbf{b}' = \mathbf{0}$ implies a restriction on $\mathbf{x}$ or $\mathbf{u}$, respectively. Both implications contradict the assumptions that the input $\mathbf{u}(t)$ can be chosen arbitrarily. If both $\mathbf{a}' = \mathbf{0}$ and $\mathbf{b}' = \mathbf{0}$ hold, $\mathbf{z}(t)$ and, thus, $\mathbf{x}(t)$ cannot be uniquely determined from eqn (3.1.9). Hence, no overall system model (3.1.1) exists. □

The second remark concerns the order of the overall system. The model (3.1.1) and (3.1.11) has been derived under the assumption (3.1.8). That is, the subsystem state spaces $\mathbf{X}_i$ are assumed to be disjoint so that the overall system state $\mathbf{x}$ is simply the collection (3.1.8) of all subsystem states. Equivalently,

$$\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2 \oplus \cdots \oplus \mathbf{X}_N \tag{3.1.20}$$

holds, where $\oplus$ denotes the direct sum of the vector spaces $\mathbf{X}_i$.

Although a model with this system state $\mathbf{x}$ exists under condition (3.1.9), this model need not be a minimal realization. Several state variables may coincide or some linear combination of them may be replaced by a single state variable. Problems with such overlapping subsystem states will be considered in connection with symmetric systems (Chapter 12), where the overlapping occurs due to the system structure, and in a generalized decomposition method (Section 3.4), where the overlapping is deliberately introduced by an expansion of the overall system state space.

3.2 HIERARCHICALLY STRUCTURED SYSTEMS

Most of the difficulties of analytical and control problems are raised by the complete interdependence of the subsystems. That is, there are links between arbitrary pairs of subsystems. Such a link from the ith to the jth subsystem need not be direct but may be mediated by one or more other subsystems.

Indirect couplings are typical of systems with sparse interconnections. They render more difficult the question of which subsystems are really coupled. The *sparsity* of interconnection means that the number of direct couplings among the subsystems is small in relation to the maximum number N^2. Sparsity must not be confused with the *weakness* of interconnections, which refers to the fact that the existing links do not severely influence the overall system performance, so that the subsystems behave similarly when coupled together or when isolated from each other.

Conceptual simplifications of analytical and control problems can be obtained if some subsystems have only a one-way effect on some others. The way in which this situation can be recognized will be investigated now.

The Interconnection Structure

The interactions among the subsystems (3.1.4) are described by the relation (3.1.6)

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{3.2.1}$$

where $\mathbf{s} = (\mathbf{s}_1' \ \mathbf{s}_2' \ \dots \ \mathbf{s}_N')'$ and $\mathbf{z} = (\mathbf{z}_1' \ \mathbf{z}_2' \ \dots \ \mathbf{z}_N')'$. The matrix $\mathbf{L}$ can be decomposed in correspondence with the vectors $\mathbf{s}$ and $\mathbf{z}$

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} & \dots & \mathbf{L}_{1N} \\ \mathbf{L}_{21} & \mathbf{L}_{22} & \dots & \mathbf{L}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{L}_{N1} & \mathbf{L}_{N2} & \dots & \mathbf{L}_{NN} \end{pmatrix}. \tag{3.2.2}$$

The block $\mathbf{L}_{ij}$ describes the couplings from the jth subsystem to the ith one. If $\mathbf{L}_{ij} = \mathbf{0}$ holds, no direct coupling exists. However, the jth subsystem may influence the ith one indirectly via other subsystems.

Under what condition this roundabout way exists can be found by a qualitative analysis of the interaction relation (3.2.1), in which only the existence of couplings rather than their strength is considered. Instead of the numeric matrix $\mathbf{L}$, the structure matrix $[\mathbf{L}]$ is used (cf. Section 2.5). $[\mathbf{L}]$ is obtained from $\mathbf{L}$ after all non-vanishing elements have been replaced by the indeterminate element '$*$'. If the interconnection signals $\mathbf{s}_i$ and $\mathbf{z}_i$ are vectors rather than scalars, $\mathbf{L}_{ij}$ in eqn (3.2.2) are matrices. The same holds for $[\mathbf{L}_{ij}]$. However, since only the existence of some interconnection should be investigated, the matrices $[\mathbf{L}_{ij}]$ will be reduced to the scalar $[[\mathbf{L}_{ij}]]$. That is, the scalar $[[\mathbf{A}]]$ is defined for an (n, m) matrix $\mathbf{A} = (a_{ij})$ by

$$[[\mathbf{A}]] = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{0} \\ * & \text{if } a_{ij} \neq 0 \text{ for at least one pair of indices } i, j. \end{cases} \tag{3.2.3}$$

For the compound matrix $\mathbf{L}$ in eqn (3.2.2), $[[\mathbf{L}]]$ is defined as the (N, N) matrix

$$[[\mathbf{L}]] = \begin{pmatrix} [[\mathbf{L}_{11}]] & [[\mathbf{L}_{12}]] & \dots & [[\mathbf{L}_{1N}]] \\ \vdots & \vdots & & \vdots \\ [[\mathbf{L}_{N1}]] & [[\mathbf{L}_{N2}]] & \dots & [[\mathbf{L}_{NN}]] \end{pmatrix}.$$

This matrix is used to describe the interconnection structure of the overall system.

An overall system with N subsystems (3.1.4) whose interconnections (3.1.6) are described by a given matrix $\mathbf{L}$ is represented by $\mathbf{S}(N, \mathbf{L})$. Then, for a given structure matrix $\mathbf{S}_1$ the class of systems (3.1.4) and (3.1.6) with structurally equivalent interactions is described by

$$\mathscr{S}_1(\mathbf{S}_1) = \{\mathbf{S}(N, \mathbf{L}): \ [[\mathbf{L}]] = \mathbf{S}_1\}. \tag{3.2.4}$$

The interconnection structure of all systems of this class can be represented by the directed graph $G(\mathbf{S}_1)$ whose N vertices visualize the subsystems and whose edges mark the direct interconnection links among the subsystems.

Example 3.1

Consider an overall system with six subsystems, $\dim \mathbf{s}_i = \dim \mathbf{z}_i = 1$ and interconnection matrix

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & l_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{25} & 0 \\ l_{31} & 0 & 0 & l_{34} & 0 & 0 \\ l_{41} & 0 & 0 & 0 & 0 & l_{46} \\ 0 & l_{52} & l_{53} & 0 & 0 & 0 \\ l_{61} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2.5}$$

The interconnections have the structure described by

$$[[\mathbf{L}]] = \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.6}$$

where $[[\mathbf{L}]] = [\mathbf{L}]$ holds since the interconnection signals are scalar. Although this matrix is sparse, it cannot be immediately recognized which subsystems are coupled in both directions. The graph $G([[\mathbf{L}]])$ with the adjacency matrix $[[\mathbf{L}]]$ from eqn (3.2.6) is shown in Figure 3.2. Obviously, the overall system consists of three groups of subsystems two of which are encircled by dashed lines. Within these groups the subsystems are strongly coupled in the sense that there are direct or indirect links between each pair of subsystems. In what follows it will be explained how these groups can be found systematically. □

Definition 3.1

Consider the class $\mathscr{S}_1$ of interconnected systems. The subsystems i and j of a system $\mathbf{S}(N, \mathbf{L}) \in \mathscr{S}_1$ are called *strongly coupled* if in the graph $G(\mathbf{S}_1)$ there exist a path from vertex i to vertex j and a path from vertex j to vertex i.

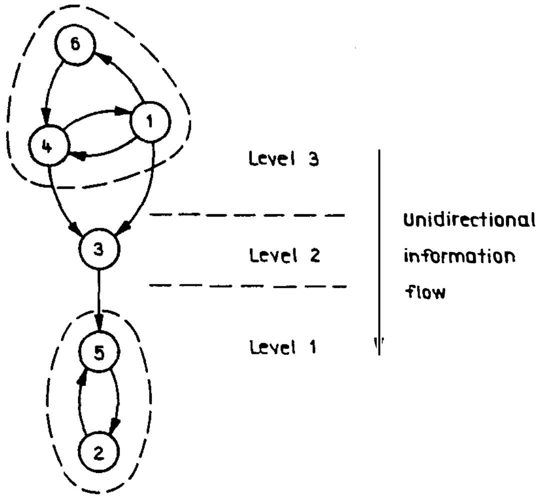

Figure 3.2 Hierarchical structure of the system in Example 3.1

That is, strongly coupled subsystems are represented by strongly connected vertices of $G(\mathbf{S}_1)$ (cf. Definition A2.1 in Appendix 2).

Decomposition of the Overall System into Groups of Strongly Coupled Subsystems

The subset of subsystems which are strongly coupled with a given subsystem i forms an equivalence class within the set of all the N subsystems of a given system $\mathbf{S}(N, \mathbf{L})$. That is, the index set

$$\mathcal{I} = \{1, 2, \ldots, N\} \tag{3.2.7}$$

which represents the numbers of the subsystems, can be uniquely decomposed into disjoint sets

$$\mathcal{I}_i = \{j_{i1}, j_{i2}, \ldots, j_{ik_i}\} \tag{3.2.8}$$

so that all pairs of subsystems of the same set $\mathcal{I}_i$ are strongly coupled whereas the subsystems of different sets $\mathcal{I}_k$, $\mathcal{I}_l$ $(k \neq l)$ do not possess this property.

Theorem 3.2

The decomposition of the overall system into strongly coupled subsystems is given by the equivalence relation on the index set $\mathcal{I}$ of the subsystems according to which $\mathscr{I}$ is decomposed into $\bar{N}$ disjoint subsets $\mathscr{I}_i$

$$\mathscr{I} = \bigcup_{i=1}^{\bar{N}} \mathscr{I}_i \qquad \mathscr{I}_i \cap \mathscr{I}_j = \emptyset \quad \text{for all } i \neq j \tag{3.2.9}$$

where all subsystems with indices of the same set $\mathscr{I}_i$ are strongly coupled with each other.

The sets $\mathscr{I}_i$ can be found by graph search algorithms. For each vertex i the set $\mathscr{R}_i$ of reachable vertices has to be determined. If $i \in \mathscr{R}_j$ and $j \in \mathscr{R}_i$ hold, then the ith and the jth subsystems belong to the same set $\mathscr{I}_k$.

The set of equivalence classes $\mathscr{I}_i$ can be renumbered in such a way that there are no interactions from subsystems of equivalence classes of lower indices towards subsystems belonging to equivalence classes of higher indices. This reordering can be represented by a permutation matrix **P**. A permutation matrix is a matrix whose only non-vanishing elements are exactly one '1' in each row and each column. The new interconnection matrix $\tilde{\mathbf{L}}$, which describes the interactions after the reordering of the subsystems, is obtained from **L** according to

$$\tilde{\mathbf{L}} = \mathbf{P}'\mathbf{L}\mathbf{P}. \tag{3.2.10}$$

The matrix $\tilde{\mathbf{L}}$ is block triangular if it is decomposed according to the decomposition (3.2.9) of the index set $\mathscr{I}$:

$$\tilde{\mathbf{L}} = \begin{pmatrix} \tilde{\mathbf{L}}_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \tilde{\mathbf{L}}_{21} & \tilde{\mathbf{L}}_{22} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{\mathbf{L}}_{\bar{N}1} & \tilde{\mathbf{L}}_{\bar{N}2} & \tilde{\mathbf{L}}_{\bar{N}3} & \dots & \tilde{\mathbf{L}}_{\bar{N}\bar{N}} \end{pmatrix}. \tag{3.2.11}$$

The diagonal blocks $\tilde{\mathbf{L}}_{ii}$ describe the couplings among those subsystems that belong to the same set $\mathscr{I}_i$ and, thus, form the ith hypersubsystem (or ith cluster of subsystems). The blocks $\tilde{\mathbf{L}}_{ij}$ describe the interconnections from subsystems of $\mathscr{I}_j$ to subsystems of $\mathscr{I}_i$.

The overall system is said to have a *hierarchical structure* since the cluster of subsystems can be grouped in different levels where the information flow is unidirectional from clusters at higher levels towards clusters at lower levels (Figure 3.2).

As a consequence, the matrix **A** of the overall system (3.1.1) is block triangular too (cf. (3.1.14) with $\tilde{\mathbf{L}}$ instead of **L**) if it is decomposed according to the clusters of subsystems

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{\bar{N}1} & \mathbf{A}_{\bar{N}2} & \dots & \mathbf{A}_{\bar{N}\bar{N}} \end{pmatrix}. \tag{3.2.12}$$